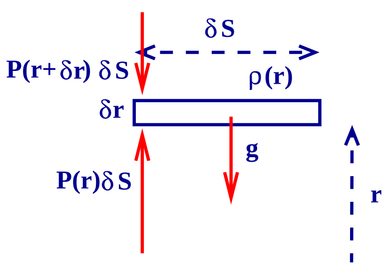
3. THE PHYSICAL STATE OF THE STELLAR INTERIOR

Fundamental assumptions:

- Although stars evolve, their properties change so slowly that at any time it is a good approximation to neglect the rate of change of these properties.
- Stars are spherical and symmetrical about their centres; all physical quantities depend just on r, the distance from the centre:

3.1 The Equation of hydrostatic equilibrium (ZG: 16-1; CO: 10.1)

> Fundamental principle 1: stars are self-gravitating bodies in dynamical equilibrium
> → balance of gravity and internal pressure forces



Consider a small volume element at a distance r from the centre, cross section δS, length δr.

$$(P_{r+\delta r} - P_r)\,\delta S + GM_r/r^2\,(\rho_r\,\delta S\,\delta r) = 0$$

$$\frac{dP_r}{dr} = -\frac{GM_r\rho_r}{r^2} \qquad (1)$$

Equation of distribution of mass:

$$M_{r+\delta r} - M_r = (dM_r/dr)\,\delta r = 4\pi r^2 \rho_r\,\delta r$$

$$\frac{dM_r}{dr} = 4\pi r^2 \rho_r \qquad (2)$$

Exercise: 3.1 Use dimensional analysis to estimate the central pressure and central temperature of a star.
– consider a point at $r = R_s/2$

$$dP_r/dr \sim -P_c/R_s \qquad \rho_r \sim \bar{\rho} = 3M_s/(4\pi R_s^3)$$

$$M_r \sim M_s/2 \qquad P_c \sim (3/8\pi)(GM_s^2/R_s^4)$$

$$(P_c)_\odot \sim 5\times10^{14}\,\mathrm{N\,m^{-2}} \quad \text{or} \quad 5\times10^{9}\,\mathrm{atm}$$

Estimate of central temperature:
Assume stellar material obeys the ideal gas equation

$$P_r = \frac{\rho_r}{\mu m_H} k T_r$$

(μ = mean molecular weight in proton masses; $\mu \sim 1/2$ for fully ionized hydrogen) and using equation (1) to obtain

$$kT_c \simeq \frac{GM_s\,\mu m_H}{R_s}$$

$$(T_c)_\odot \sim 2\times10^7\,\mathrm{K} \quad \bar{\rho}_\odot \sim 1.4\times10^3\,\mathrm{kg\,m^{-3}} \quad (\text{c.f. } (T_s)_\odot \sim 5800\ \mathrm{K})$$

- Although the Sun has a mean density similar to that of water, the high temperature requires that it should be gaseous throughout.
- the average kinetic energy of the particles is higher than the binding energy of atomic hydrogen so the material will be highly ionized, i.e is a plasma.

3.2 The Dynamical timescale (ZG: P5-4; CO: 10.4): t_D

- Time for star to collapse completely if pressure forces were negligible ($\delta M \ddot{r} = -\delta M\, g$)

$$(\rho\ \delta S \delta r)\, \ddot{r} = -(GM_r/r^2)\,(\rho\ \delta S \delta r)$$

- Inward displacement of element after time t is given by

$$s = (1/2)\, gt^2 = (1/2)\,(GM_r/r^2)\, t^2$$

- For estimate of t_{dyn}, put $s \sim R_s, r \sim R_s, M_r \sim M_s$; hence

$$t_{dyn} \sim (2R_s^3/GM_s)^{1/2} \sim \{3/(2\pi G\bar{\rho})\}^{1/2}$$

$$(t_{dyn})_\odot \sim 2300\ s \sim 40\ mins$$

Stars adjust very quickly to maintain a balance between pressure and gravitational forces.
General rule of thumb: $t_{dyn} \simeq 1/\sqrt{4G\bar{\rho}}$

3.3 The virial theorem (ZG: P5-2; CO: 2.4)

$$dP_r/dr = -GM_r\rho_r/r^2$$

$$4\pi r^3 dP_r = -(GM_r/r) 4\pi r^2 \rho_r dr$$

$$4\pi[r^3 P_r]_{r=0, P=P_c}^{r=R_s, P=P_s} - 3\int_0^{R_s} P_r\, 4\pi r^2 dr = -\int_0^{R_s} (GM_r/r) 4\pi r^2 \rho_r dr$$

$$\int_0^{R_s} 3P_r\, 4\pi r^2 dr = \int_0^{R_s} (GM_r/r) 4\pi r^2 \rho_r dr$$

Thermal energy/unit volume $u = nfkT/2 = (\rho/\mu m_H) fkT/2$
Ratio of specific heats $\gamma = c_p/c_v = (f+2)/f$ $(f = 3 : \gamma = 5/3)$

$$u = \{1/(\gamma - 1)\}(\rho kT/\mu m_H) = P/(\gamma - 1)$$

$$\boxed{3(\gamma - 1)U + \Omega = 0}$$

U = total thermal energy; Ω = total gravitational energy.
For a fully ionized, ideal gas $\gamma = 5/3$ and $2U + \Omega = 0$
Total energy of star $E = U + \Omega$

$$\boxed{E = -U = \Omega/2}$$

Note: E is negative and equal to $\Omega/2$ or $-U$. A decrease in E leads to a decrease in Ω but an increase in U and hence T. A star, with no hidden energy sources, composed of a perfect gas contracts and heats up as it radiates energy.

> Fundamental principle 2: stars have a negative 'heat capacity', they heat up when their total energy decreases

Important implications of the virial theorem:

- stars become hotter when their total energy decreases (→ normal stars contract and heat up when there is no nuclear energy source because of energy losses from the surface);
- nuclear burning is self-regulating in non-degenerate cores: e.g. a sudden increase in nuclear burning causes expansion and cooling of the core: negative feedback → stable nuclear burning.

3.4 Sources of stellar energy: (CO: 10.3)

> Fundamental principle 3: since stars lose energy by radiation, stars supported by thermal pressure require an energy source to avoid collapse

Provided stellar material always behaves as a perfect gas, thermal energy of star ∼ gravitational energy.

- total energy available $\sim GM_s^2/2R_s$
- thermal time-scale (Kelvin-Helmholtz timescale, the timescale on which a star radiates away its thermal energy)):
  $t_{th} \sim GM_s^2/(2R_sL_s)$
  $(t_{th})_\odot \sim 0.5 \times 10^{15}$ sec $\sim 1.5 \times 10^7$ years.
- e.g. the Sun radiates $L_\odot \sim 4 \times 10^{26}$ W, and from geological evidence $L_\odot$ has not changed significantly over $t \sim 10^9$ years

The thermal and gravitational energies of the Sun are not sufficient to cover radiative losses for the total solar lifetime.
Only nuclear energy can account for the observed luminosities and lifetimes of stars

- Largest possible mass defect available when H is transmuted into Fe: energy released = 0.008 × total mass. For the Sun $(E_N)_\odot = 0.008\, M_\odot c^2 \sim 10^{45}$ J
- Nuclear timescale $(t_N)_\odot \sim (E_N)_\odot / L_\odot \sim 10^{11}$ yr
- Energy loss at stellar surface as measured by the stellar luminosity is compensated by energy release from nuclear reactions throughout the stellar interior.

$$L_s = \int_0^{R_s} \varepsilon_r \rho_r \, 4\pi r^2 dr$$

ε_r is the nuclear energy released per unit mass per sec and will depend on T_r, ρ_r and composition

$$\boxed{\frac{dL_r}{dr} = 4\pi r^2 \rho_r \varepsilon_r \qquad (3)}$$

for any elementary shell.

- During rapid evolutionary phases, (i.e. $t \ll t_{th}$)

$$\boxed{\frac{dL_r}{dr} = 4\pi r^2 \rho_r \left(\varepsilon_r - T\frac{dS}{dt}\right) \qquad (3a),}$$

where $-TdS/dt$ is called a gravitational energy term.

SUMMARY III: STELLAR TIMESCALES

- dynamical timescale: $t_{dyn} \simeq \dfrac{1}{\sqrt{4G\rho}}$
  $\sim 30\,\text{min}\,\left(\rho/1000\,\text{kg}\,\text{m}^{-3}\right)^{-1/2}$
- thermal timescale (Kelvin-Helmholtz): $t_{KH} \simeq \dfrac{GM^2}{2RL}$
  $\sim 1.5 \times 10^7\,\text{yr}\,(M/M_\odot)^2\,(R/R_\odot)^{-1}\,(L/L_\odot)^{-1}$
- nuclear timescale: $t_{nuc} \simeq \underbrace{M_c/M}_{\text{core mass}} \; \underbrace{\eta}_{\text{efficiency}} \; (Mc^2)/L$
  $\sim 10^{10}\,\text{yr}\,(M/M_\odot)^{-3}$

3.5 Energy transport (ZG: P5-10, 16-1, CO: 10.4)

The size of the energy flux is determined by the mechanism that provides the energy transport: conduction, convection or radiation. For all these mechanisms the temperature gradient determines the flux.

- Conduction does not contribute seriously to energy transport through the interior
  - ▷ At high gas density, mean free path for particles << mean free path for photons.
  - ▷ Special case, degenerate matter – very effective conduction by electrons.
- The thermal radiation field in the interior of a star consists mainly of X-ray photons in thermal equilibrium with particles.
- Stellar material is opaque to X-rays (bound-free absorption by inner electrons)
- mean free path for X-rays in solar interior ~ 1 cm.
- Photons reach the surface by a "random walk" process and as a result of many interactions with matter are degraded from X-ray to optical frequencies.
- After N steps of size l, the distribution has spread to $\simeq \sqrt{N}\, l$. For a photon to "random walk" a distance R_s, requires a diffusion time (in steps of size l)

$$t_{\rm diff} = N \times \frac{l}{c} \simeq \frac{R_s^2}{lc}$$

For $l = 1$ cm, $R_s \sim R_\odot \rightarrow t_{\rm diff} \sim 5 \times 10^3$ yr.

Energy transport by radiation:

- Consider a spherical shell of area $A = 4\pi r^2$, at radius r of thickness dr.
- radiation pressure

$$P_{\rm rad} = \frac{1}{3} a T^4 \qquad (i)$$

(=momentum flux)

- rate of deposition of momentum in region $r \rightarrow r + dr$

$$-\frac{dP_{\rm rad}}{dr}\, dr\, 4\pi r^2 \qquad (ii)$$

- define opacity κ [m^2/kg], so that fractional intensity loss in a beam of radiation is given by

$$\frac{dI}{I} = -\kappa\rho\, dx,$$

where ρ is the mass density and

$$\tau \equiv \int \kappa\rho dx$$

is called optical depth (note: $I = I_0 \exp(-\tau)$)

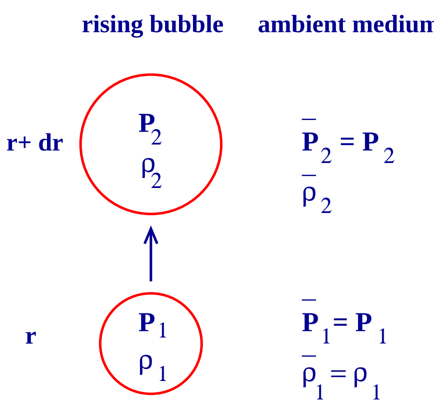
  - ▷ $1/\kappa\rho$: mean free path
  - ▷ $\tau \gg 1$: optically thick
  - ▷ $\tau \ll 1$: optically thin

- rate of momentum absorption in shell $L(r)/c\, \kappa\rho\, dr$. Equating this with equation (ii) and using (i):

$$L_r = -4\pi r^2 \frac{4ac}{3\kappa\rho} T^3 \frac{dT}{dr} \qquad (4a)$$

Energy transport by convection:

- Convection occurs in liquids and gases when the temperature gradient exceeds some typical value.
- Criterion for stability against convection (Schwarzschild criterion)



▷ consider a bubble with initial ρ_1, P_1 rising by an amount dr, where the ambient pressure and density are given by $\overline{\rho(r)}, \overline{P(r)}$.

▷ the bubble expands adiabatically, i.e $P_2 = P_1 \left(\frac{\rho_2}{\rho_1}\right)^{\gamma}$ (γ = adiabatic exponent)

▷ assuming the bubble remains in pressure equilibrium with the ambient medium, i.e. $P_2 = \overline{P_2} = \overline{P(r+dr)} \simeq P_1 + (dP/dr)\, dr$,

$$\rho_2 = \rho_1 \left(\frac{P_2}{P_1}\right)^{1/\gamma} \simeq \rho_1 \left(1 + \frac{1}{P}\frac{dP}{dr}\, dr\right)^{1/\gamma}$$
$$\simeq \rho_1 + \frac{\rho_1}{\gamma P}\frac{dP}{dr}\, dr$$

▷ convective stability if $\rho_2 - \overline{\rho_2} > 0$ (bubble will sink back)

$$\boxed{\frac{\rho}{\gamma P}\frac{dP}{dr} - \frac{d\rho}{dr} > 0}$$

- For a perfect gas (negligible radiation pressure)

$$P = \rho k T / (\mu m_H)$$

- Provided μ does not vary with position (no changes in ionization or dissociation)

$$-[1 - (1/\gamma)](T/P)\, dP/dr > -dT/dr \text{ (both negative)}$$

- or magnitude of adiabatic dT/dr > magnitude of actual dT/dr.
- Alternatively, $\frac{P}{T}\frac{dT}{dP} < \frac{\gamma - 1}{\gamma}$
- There is no generally accepted theory of convective energy transport at present. The stability criterion must be checked at every layer within a stellar model: dP/dr from equation (1) and dT/dr from equation (4). The stability criterion can be broken in two ways:

1. Large opacities or very centrally concentrated nuclear burning can lead to high (unstable) temperature gradients e.g. in stellar cores.
2. $(\gamma - 1)$ can be much smaller than 2/3 for a monatomic gas, e.g. in hydrogen ionization zones.

Influence of convection

(a) Motions are turbulent: too slow to disturb hydrostatic equilibrium.
(b) Highly efficient energy transport: high thermal energy content of particles in stellar interior.
(c) Turbulent mixing so fast that composition of convective region homogeneous at all times.
(d) Actual dT/dr only exceeds adiabatic dT/dr by very slight amount.

Hence to sufficient accuracy (in convective regions)

$$\frac{dT}{dr} = \frac{\gamma - 1}{\gamma}\frac{T}{P}\frac{dP}{dr} \qquad (4b)$$

This is not a good approximation close to the surface (in particular for giants) where the density changes rapidly.