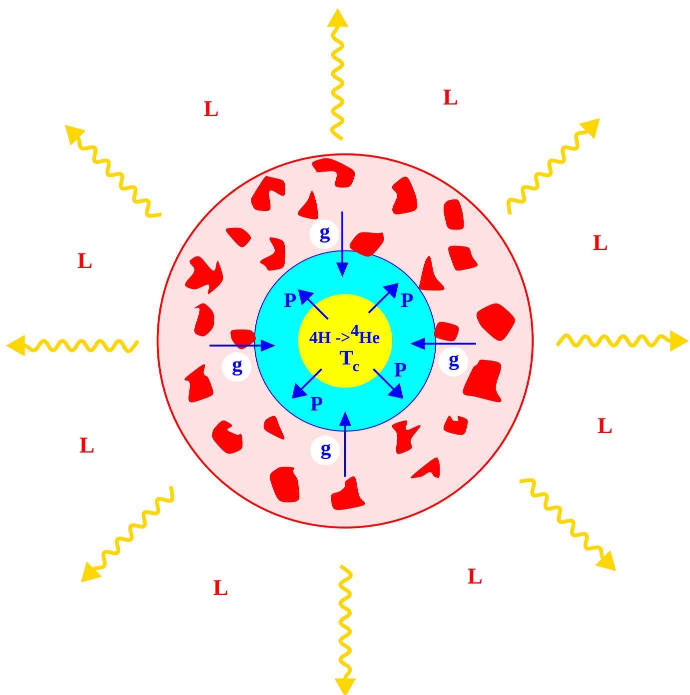



## SUMMARY IV: FUNDAMENTAL PRINCIPLES

- Stars are self-gravitating bodies in dynamical equilibrium → balance of gravity and internal pressure forces (hydrostatic equilibrium);
- stars lose energy by radiation from the surface → stars supported by thermal pressure require an energy source to avoid collapse, e.g. nuclear energy, gravitational energy (energy equation);
- the temperature structure is largely determined by the mechanisms by which energy is transported from the core to the surface, radiation, convection, conduction (energy transport equation);
- the central temperature is determined by the characteristic temperature for the appropriate nuclear fusion reactions (e.g. H-burning: $10^7$ K; He-burning: $10^8$ K);
- normal stars have a negative 'heat capacity' (virial theorem): they heat up when their total energy decreases (→ normal stars contract and heat up when there is no nuclear energy source);
- nuclear burning is self-regulating in non-degenerate cores (virial theorem): e.g. a sudden increase in nuclear burning causes expansion and cooling of the core: negative feedback → stable nuclear burning;
- the global structure of a star is determined by the simultaneous satisfaction of these principles → the local properties of a star are determined by the global structure.
  (Mathematically: it requires the simultaneous solution of a set of coupled, non-linear differential equations with mixed boundary conditions.)

## 4 THE EQUATIONS OF STELLAR STRUCTURE

In the absence of convection:

$$\frac{dP_r}{dr} = \frac{-GM_r\rho_r}{r^2} \quad (1)$$

$$\frac{dM_r}{dr} = 4\pi r^2\rho_r \quad (2)$$

$$\frac{dL_r}{dr} = 4\pi r^2\rho_r\left(\varepsilon_r - T\frac{dS}{dt}\right) \quad (3)$$

$$\frac{dT_r}{dr} = \frac{-3\kappa_r L_r\rho_r}{16\pi acr^2T_r^3} \quad (4a)$$

### 4.1 The Mathematical Problem (GZ: 16-2; CO: 10.5)

- $P_r, \kappa_r, \varepsilon_r$ are functions of $\rho, T$, chemical composition
- Basic physics can provide expressions for these.
- In total, there are four, coupled, non-linear, partial differential equations (+ three constitutive relations) for seven unknowns: $P, \rho, T, M, L, \kappa, \varepsilon$ as functions of $r$.
- These completely determine the structure of a star of given composition subject to boundary conditions.
- In general, only numerical solutions can be obtained (i.e. computer).
- Four (mixed) boundary conditions needed:
  - ▷ at centre: $M_r = 0$ and $L_r = 0$ at $r = 0$ (exact)
  - ▷ at surface: $L_s = 4\pi R_s^2\sigma T_{eff}^4$ (blackbody relation)
    (surface = photosphere, where $\tau \simeq 1$)
    $P = (2/3)\, g/\kappa$ (atmosphere model)

  (sometimes: $P(R_s) = 0$ [rough], but *not* $T(R_s) = 0$)

#### 4.1.1 Uniqueness of solution: the Vogt Russell "Theorem" (CO: 10.5)

> "For a given chemical composition, only a single equilibrium configuration exists for each mass; thus the internal structure of the star is fixed."

- This "theorem" has not been proven and is not even rigorously true; there are known exceptions

#### 4.1.2 The equilibrium solution and stellar evolution:

- If there is no bulk motions in the interior of a star (i.e. no convection), changes of chemical composition are localised in regions of nuclear burning The structure equations (1) to (4) can be supplemented by equations of the type:

$$\partial/\partial t\,(\text{composition})_M = f(\rho, T, \text{composition})$$

- Knowing the composition as a function of M at a time $t_0$ we can solve (1) to (4) for the structure at $t_0$. Then

$$(\text{composition})_{M,t_0+\delta t} = (\text{composition})_{M,t_0} + \partial/\partial t\,(\text{composition})_M\,\delta t$$

- Calculate modified structure for new composition and repeat to discover how star evolves (not valid if stellar properties change so rapidly that time dependent terms in (1) to (4) cannot be ignored).

#### 4.1.3 Convective regions: (GZ: 16-1; CO: 10.4)

- Equations (1) to (3) unchanged.
- for efficient convection (neutral buoyancy):

$$\frac{P}{T}\frac{dT}{dP} = \frac{\gamma - 1}{\gamma} \quad (4b)$$

- $L_{rad}$ is calculated from equation (4) once the above

## 4.2 THE EQUATION OF STATE

### 4.2.1 Perfect gas: (GZ: 16-1: CO: 10.2)

$$P = NkT = \frac{\rho}{\mu m_H} kT$$

N is the number density of particles; $\mu$ is the mean particle mass in units of $m_H$. Define:

X = mass fraction of hydrogen (Sun: 0.70)

Y = mass fraction of helium (Sun: 0.28)

Z = mass fraction of heavier elements (metals) (Sun: 0.02)

- $X + Y + Z = 1$
- If the material is assumed to be fully ionized:

| Element | No. of atoms | No. of electrons |
|---|---|---|
| Hydrogen | $X\rho/m_H$ | $X\rho/m_H$ |
| Helium | $Y\rho/4m_H$ | $2Y\rho/4m_H$ |
| Metals | $[Z\rho/(Am_H)]$ | $(1/2)AZ\rho/(Am_H)$ |

- A represents the average atomic weight of heavier elements; each metal atom contributes $\sim A/2$ electrons.
- Total number density of particles:

  $N = (2X + 3Y/4 + Z/2)\,\rho/m_H$

  ▷ $(1/\mu) = 2\,X + 3/4\,Y + 1/2\,Z$
- This is a good approximation to $\mu$ except in cool, outer regions.
- When Z is negligible: $Y = 1 - X$; $\mu = 4/(3 + 5X)$
- Inclusion of radiation pressure in P:

$$P = \rho kT/(\mu m_H) + aT^4/3.$$

  (important for massive stars)

### 4.2.2 Degenerate gas: (GZ: 17-1; CO: 15.3)

- First deviation from perfect gas law in stellar interior occurs when electrons become degenerate.
- The number density of electrons in phase space is limited by the Pauli exclusion principle.

$$n_e\, dp_x dp_y dp_z\, dxdydz \leq (2/h^3)\, dp_x dp_y dp_z\, dxdydz$$

- In a completely degenerate gas all cells for momenta smaller than a threshold momentum $p_0$ are completely filled (Fermi momentum).
- The number density of electrons within a sphere of radius $p_0$ in momentum space is (at $T = 0$):

$$N_e = \int_0^{p_0} (2/h^3)\, 4\pi p^2 dp = (2/h^3)(4\pi/3)p_0^3$$

- From kinetic theory

$$P_e = (1/3) \int_0^{\infty} p\, v(p) n(p) dp$$

(a) Non-relativistic complete degeneracy:

$$v(p) = p/m_e \qquad \text{for all } p$$

$$P_e = (1/3) \int_0^{p_0} (p^2/m_e)(2/h^3)\, 4\pi p^2\, dp$$

$$= \{8\pi/(15 m_e h^3)\} p_0^5 = \{h^2/(20 m_e)\}(3/\pi)^{2/3}\, N_e^{5/3}.$$

(b) Relativistic complete degeneracy:

$$\mathbf{v(p) \sim c}$$

$$\mathbf{P_e = (1/3)\int_0^{p_0} pc(2/h^3)\, 4\pi p^2\, dp}$$

$$\mathbf{= (8\pi c/3h^3) p_0^4/4 = (2\pi c/3h^3)\, p_0^4}$$

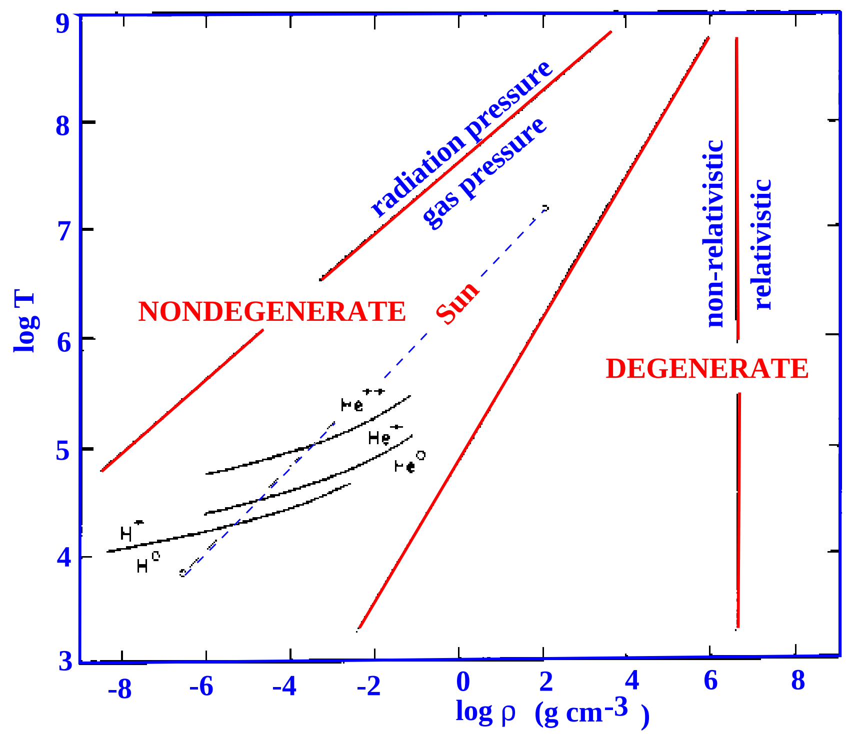
$$\mathbf{= (hc/8)(3/\pi)^{1/3}\, N_e^{4/3}.}$$

- Also $\mathbf{N_e = (X + Y/2 + Z/2)\,\rho/m_H = (1/2)(1+X)\,\rho/m_H}$.
- For intermediate regions use the full relativistic expression for $\mathbf{v(p)}$.
- For ions we may continue to use the non-degenerate equation:
- $\mathbf{P_{ions} = (1/\mu_{ions})(\rho k T/m_H)}$ where $\mathbf{(1/\mu_{ions}) = X + Y/4}$.

Conditions where degeneracy is important:

(a) Non-relativistic – interiors of white dwarfs; degenerate cores of red giants.

(b) Relativistic - very high densities only; interiors of white dwarfs.




Temperature-density diagram for the equation of state (Schwarzschild 1958)

## 4.3 THE OPACITY (GZ: 10-2; CO: 9.2)

The rate at which energy flows by radiative transfer is determined by the opacity (cross section per unit mass $[\mathrm{m}^2/\mathrm{kg}]$)

$$\mathrm{dT/dr} = -3\kappa \mathrm{L}\rho/(16\pi \mathrm{acr}^2\mathrm{T}^3) \qquad (4)$$

In degenerate stars a similar equation applies with the opacity representing resistance to energy transfer by electron conduction.

Sources of stellar opacity:

1. bound-bound absorption (negligible in interiors)

2. bound-free absorption

3. free-free absorption

4. scattering by free electrons

- usually use a mean opacity averaged over frequency, Rosseland mean opacity (see textbooks).

Approximate analytical forms for opacity:

High temperature: $\kappa = \kappa_1 = 0.020\,\mathrm{m}^2\,\mathrm{kg}^{-1}\,(1 + \mathrm{X})$

Intermediate temperature: $\kappa = \kappa_2 \rho \mathrm{T}^{-3.5}$ (Kramer's law)

Low temperature: $\kappa = \kappa_3 \rho^{1/2} \mathrm{T}^4$

- $\kappa_1, \kappa_2, \kappa_3$ are constant for stars of given chemical composition but all depend on composition.

Opacities