# Astrophysics Graduate Course

## Computational Problem: Models of White Dwarfs and the Chandrasekhar Mass

### (Ph. Podsiadlowski, Oxford, MT05)

The purpose of this problem is to develop simple numerical models for the structure of white-dwarf stars and to estimate the Chandrasekhar mass, i.e. the maximum mass of a white dwarf[1].

Section 1 describes a simple approximation to the equation of state, $P(\rho)$, for a completely degenerate, but arbitrarily relativistic electron gas. Note that the gas is taken to have a negligibly small temperature so that thermal contributions to the pressure are unimportant.

Section 2 presents a derivation of the basic equation of a polytrope (leading to a Lane-Emden-type equation).

Section 3 sets up the differential equation for the white dwarf case which is similar to a Lane-Emden equation and guides you through the basic steps to solve this equation numerically.

Section 4 asks to plot various results of interest.

For those of you who have little experience with programming, Section 5 lists some Fortran programs that solve most of the problem.

# 1 The equation of state at low temperatures

## 1.1 Non-relativistic Fermi pressure

$$\begin{aligned} P_{\rm N} &= \frac{h^2}{20\, m_e} \left(\frac{3}{\pi}\right)^{2/3} n_e^{5/3} \\ &= \frac{h^2}{20\, m_e} \left(\frac{3}{\pi}\right)^{2/3} \frac{1}{(2m_p)^{5/3}}\, \rho^{5/3} \end{aligned} \tag{1}$$

(appropriate for He, C or O, where $\rho$ is the total mass density).

$$P_{\rm N} = K_{\rm N}\, \rho^{5/3},\ \text{where } K_{\rm N} = 3.166 \times 10^{12} \text{ (cgs units)}. \tag{2}$$

## 1.2 Extreme relativistic Fermi pressure

$$\begin{aligned} P_{\rm R} &= \frac{hc}{8} \left(\frac{3}{\pi}\right)^{1/3} n_e^{4/3} \\ &= \frac{hc}{8} \left(\frac{3}{\pi}\right)^{1/3} \frac{1}{(2m_p)^{4/3}}\, \rho^{4/3} \quad \text{(for He, C, or O)}. \end{aligned} \tag{3}$$

[1]The setup is a close adoption of a problem set up by Saul Rappaport at M.I.T.

$$P_{\mathrm{R}} = K_{\mathrm{R}}\,\rho^{4/3}, \text{ where } K_{\mathrm{R}} = 4.936 \times 10^{14} \text{ (cgs units)}. \tag{4}$$

The characteristic density $\rho_0$ which separates the two regimes can be obtained by equating equations 2 and 4, which leads to

$$\rho_0 = \left(\frac{K_{\mathrm{R}}}{K_{\mathrm{N}}}\right)^3 = 3.789 \times 10^6\,\mathrm{g\,cm^{-3}}.$$

To obtain an approximate expression that combines both limits, one can define

$$P \equiv \frac{P_{\mathrm{N}}\,P_{\mathrm{R}}}{\sqrt{P_{\mathrm{N}}^2 + P_{\mathrm{R}}^2}}.$$

*Show that this definition leads to*

$$P = \frac{K_{\mathrm{N}}\,\rho^{5/3}}{\left[1 + (\rho/\rho_0)^{2/3}\right]^{1/2}}.$$

## 2 Structure equation

Combine the equation of hydrostatic equilibrium

$$\boldsymbol{\nabla} P = \mathbf{g}\,\rho,$$

and Poisson's equation

$$\boldsymbol{\nabla}\cdot\mathbf{g} = -4\pi G\rho$$

to obtain the equation for a spherical polytrope

$$\boxed{\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r}\right) = -4\pi G\rho.}$$

## 3 The numerical problem

a) Let

$$\rho(r) = \rho_0\,\theta(r),$$

where $\rho_0$ is the "matching" density defined above. Note that the pressure now takes the form:

$$P = K_{\mathrm{N}}\,\rho_0^{5/3}\,\theta^{5/3}\,(1 + \theta^{2/3})^{-1/2}.$$

b) Let

$$r = s\,a,$$

where $s$ is a dimensionless radial distance and $a$ is a length scale-factor to be formed from the constants $K_{\mathrm{N}}$, $\rho_0$ and $G$ (see next step).

c) Write out the dimensionless differential equation for $\theta(s)$ and show that this implies $a = 1.557 \times 10^8\,\mathrm{cm}$.

[Answer: $\dfrac{1}{s^2}\dfrac{\mathrm{d}}{\mathrm{d}s}\left[s^2\,\dfrac{1}{\theta}\,\dfrac{\mathrm{d}\left(\theta^{5/3}\,(1+\theta^{2/3})^{-1/2}\right)}{\mathrm{d}s}\right] = -\theta.$]

d) Reduce the second-order differential equation to two coupled, first-order equations by defining:

$$V = \frac{\mathrm{d}\theta}{\mathrm{d}s}.$$

The equations are then of the form:

$$\begin{aligned} \frac{\mathrm{d}\theta}{\mathrm{d}s} &= V, \\ \frac{\mathrm{d}V}{\mathrm{d}s} &= f(V,\theta,s). \end{aligned} \tag{5}$$

[Answer:

$$f(V,\theta,s) = -\frac{2}{s}\,V - \frac{C}{B}\,V^2 - \frac{1}{B}\,\theta$$

where

$$\begin{aligned} B &= (5/3)\,\theta^{-1/3}\,(1+\theta^{2/3})^{-1/2} - (1/3)\,\theta^{1/3}\,(1+\theta^{2/3})^{-3/2} \quad \text{and} \\ C &= -(5/9)\,\theta^{-4/3}\,(1+\theta^{2/3})^{-1/2} - (2/3)\,\theta^{-2/3}\,(1+\theta^{2/3})^{-3/2} + (1/3)\,(1+\theta^{2/3})^{-5/2}. \end{aligned}$$ ]

e) Start the integration from the centre of the star with the boundary conditions:

$$\theta(0) = \theta_{\mathrm{c}}, \quad \left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)_0 = 0. \tag{6}$$

Note that $\theta_{\mathrm{c}}$ is the central density in units of $\rho_0$.

f) Set up a 4th-order Runge-Kutta algorithm (or other equivalent integration scheme) to integrate these first-order coupled equations.

g) Integrate the coupled differential equations until the density $\theta$ goes to zero. This defines the surface of the star. To avoid numerical difficulties, stop the integration when $\theta/\theta_{\mathrm{c}}$ reaches $\sim 0.1\,\%$.

h) As you are integrating the stellar structure equation, keep track of the mass of the star, i.e. do the integral:

$$m = \int_0^{s_{\max}} \theta(s)\,s^2\,\mathrm{d}s,$$

where $m$ is the dimensionless mass of the star. The actual mass of the star is then given by

$$M = 4\pi\rho_0 a^3 m = 0.090\,m\,M_\odot,$$

and the radius is given by

$$R = s_{\max} a.$$

i) Repeat the calculation for a range of densities covering the interval $10^4\,\mathrm{g\,cm^{-3}}$ to $10^{12}\,\mathrm{g\,cm^{-3}}$. A few values of $\theta_{\mathrm{c}}$ per logarithmic decade will be sufficient (i.e. you may need between 15 and 30 white-dwarf models).

# 4 Results

a) Make a plot of the masses of your white-dwarf models (in $M_\odot$) vs. the logarithm of their central density. Identify the maximum stable mass of a white dwarf.

b) Make a plot of the radii of your white-dwarf models (in units of $R_\odot/100$) vs. the logarithm of their central density.

c) Make a plot of the radii of your white-dwarf models (in units of $R_\odot/100$) vs. their mass (in $M_\odot$).

d) For the $1\,M_\odot$ and $1.3\,M_\odot$ models, plot the density vs. radial distance from the stellar centre.

# 5 Fortran programs

The following 3 programs/subroutines (available from my website) solve the basic problem.

- Program "dwarf.f" determines the mass and radius of a white dwarf for a given central density (provided as input).
- Subroutine "mers.f" contains a generic Runge-Kutta-Merson integrator (with stepsize control) to solve a system of $N$ coupled, first-order ordinary differential equations.
- Subroutine "ext.f" performs the function evaluations for the white-dwarf problem (i.e. the right-hand sides of equations (5)).

You may use or modify these subroutines to solve the basic numerical problem, but you probably want to make it more efficient by changing the handling of the input and the output for the subsequent plotting.