

THIRD-YEAR ASTROPHYSICS

Problem Set: Stellar Structure and Evolution

(Dr Ph. Podsiadlowski, Michaelmas Term 2006)

1. Measuring Stellar Parameters.

Sirius is a visual binary with a period of 49.94 yr. Its measured trigonometric parallax is $0.377''$ and, assuming that the plane of the orbit is in the plane of the sky, the true, deprojected angular extent of the semimajor axis of the binary is $7.62''$. The ratio of the distances of Sirius A and Sirius B from the centre of mass is $a_A/a_B = 0.466$.

- Find the mass of each member of the system.
- The absolute bolometric magnitude of Sirius A is 1.33, and Sirius B has an absolute bolometric magnitude of 8.57. Determine their luminosities. Express your answers in terms of solar luminosities.
- The effective temperature of Sirius B is estimated to be approximately 27,000 K. Estimate its radius and compare your answer to the radii of the Sun and Earth.

[Based on Problem 7.4 in Carroll & Ostlie, Modern Astrophysics]

2. The Central Pressure.

By integrating the equation of hydrostatic equilibrium in the form

$$\frac{dP_r}{dM_r} = -\frac{GM_r}{4\pi r^4},$$

where M_r denotes the mass enclosed at radius r and P_r is the pressure at this radius (mass coordinate), show that a rigorous lower limit for the central pressure, P_c , can be obtained as

$$P_c > \frac{GM^2}{8\pi R^4},$$

where M and R are the total mass and radius of the star, respectively. Estimate the minimum central pressure for the Sun ($1 M_\odot = 2 \times 10^{30}$ kg and $1 R_\odot = 7 \times 10^8$ m).

What is the dynamical timescale of a star? Compare the dynamical timescale of the Sun with the timescale for hydrogen burning. Assume that the Sun remains on the main sequence at a constant luminosity of 3.9×10^{26} W and that hydrogen is converted into helium in the innermost 10 per cent of the mass of the Sun.

3. The Schwarzschild Criterion for Convection.

Derive the Schwarzschild criterion for convective instability. Using this criterion, explain briefly why stars on the upper main sequence have convective cores and stars on the lower main sequence have convective envelopes.

The material in the envelope of a star has a ratio of specific heat capacities $\gamma = \frac{4}{3}$, and the star is sufficiently centrally condensed that the mass in the envelope is negligible compared to the core mass, M . The envelope is convectively unstable in such a way that the Schwarzschild criterion is just marginally satisfied. Assuming that the gas obeys the ideal gas law, show that the temperature within the envelope varies with radius r as

$$T = \frac{GM\mu m_{\text{H}}}{4k} \left(\frac{1}{r} - \frac{1}{r_{\text{s}}} \right) + T_{\text{s}},$$

where μ is the mean molecular weight of the stellar material, m_{H} the mass of the hydrogen atom, k the Boltzmann constant, G the gravitational constant, r_{s} the surface radius and T_{s} the surface temperature of the star.

4. The Virial Theorem.

Starting from the equation of hydrostatic equilibrium, show that the relation between the total thermal energy, U , and the total gravitational energy, Ω , of a star is given by

$$3(\gamma - 1)U + \Omega = 0$$

where γ is a mean adiabatic exponent of the stellar material (i.e. the ratio of specific heats for an ideal gas).

For a fully, ionized, ideal gas, $\gamma = 5/3$. Explain why this implies that stars effectively have a negative heat capacity.

Use this fact to explain

- (a) why nuclear burning in stars like the Sun is stable (i.e. does not lead to a nuclear runaway);
- (b) the basic principles governing the evolution of stars in phases when nuclear burning is not important (from the pre-main-sequence phase to the supernova stage).

For stars supported mainly by radiation pressure, $\gamma \rightarrow 4/3$. Show that this implies that their total energy goes to zero. Discuss the implication of this result in terms of the stability of such stars.

5. Mass-Luminosity Relations.

Using the condition of hydrostatic equilibrium, show that for a star which has an equation of state given by the ideal gas law, its central temperature, T_{c} , can be estimated as

$$kT_{\text{c}} \simeq \frac{GM\mu m_{\text{H}}}{R},$$

where M is the mass of the star, R its radius and μ the mean molecular weight of the stellar material [$\mu \simeq 0.6$ for a star like the Sun]. Estimate T_c for the Sun and comment on what the result implies for the energy production in the centre of the Sun.

Use this result and the equation of radiative transfer to show that the mass–luminosity relation for low- and intermediate-mass stars can be written as

$$L \propto \frac{\mu^4 M^3}{\langle \kappa \rangle},$$

where L is the star’s luminosity and $\langle \kappa \rangle$ is a characteristic average opacity of the stellar material. For intermediate-mass stars, the opacity is dominated by Thomson scattering and is approximately constant, $\kappa = \kappa_{\text{Th}}$ (hence L scales with M^3). For low-mass stars, the opacity can be approximated by Kramer’s law, $\kappa = \kappa_0 \rho T^{-3.5}$. Obtain the scaling of the luminosity with mass for this case.

6. The Coulomb Barrier.

Estimate the height of the Coulomb barrier between two protons and compare this with the mean thermal energy of a proton at a temperature $T \approx 10^7$ K. Explain how thermonuclear reactions can take place in stellar interiors at such temperatures. Outline the main reactions involved in the conversion of hydrogen into helium in a low-mass star such as the Sun.

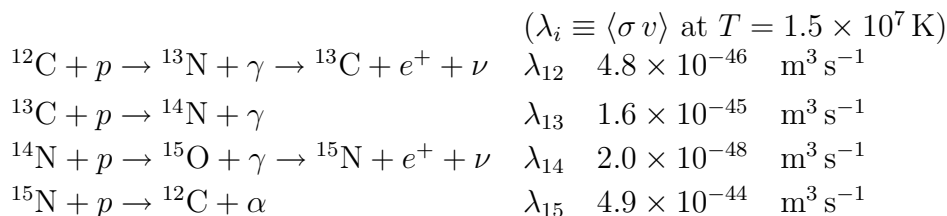
The thermonuclear rate at which protons are destroyed by the proton-proton reaction at stellar temperatures is given approximately by

$$R_{\text{pp}} = 1.1 \times 10^{15} \rho^2 X_{\text{H}}^2 T^{-2/3} \exp(-3381 (T/\text{K})^{-1/3}) \text{m}^{-3} \text{s}^{-1},$$

where ρ is the density of stellar material and X_{H} is the mass fraction of hydrogen. Estimate the lifetime (in years) of protons against the p-p reaction in material where $T = 1.5 \times 10^7$ K, $\rho = 10^5 \text{kg m}^{-3}$ and $X_{\text{H}} = 0.5$. Comment on your result.

7. The CN Cycle in the Sun [*Harder*].

The CN-cycle consists of the following reaction network



- a) Write down the differential equations which govern the abundance changes for ^{12}C , ^{13}C , ^{14}N , ^{15}N (assume that the inverse beta decays occur instantaneously).

[Hint: One of the equations can be written as $dn_{12}/dt = -n_{12} n_p \lambda_{12} + n_{15} n_p \lambda_{15}$, where n_{12} is the number density of ^{12}C , etc.]

- b) Assume that the CN-cycle is in equilibrium in the core of the Sun ($T_{\text{core}} \simeq 1.5 \times 10^7$ K). Taking the initial mass fractions of the CN nuclei to have been $X_{12}^0 = 4 \times 10^{-3}$, $X_{13}^0 = X_{14}^0 = X_{15}^0 = 0$, determine the present equilibrium mass fractions of X_{12} , X_{13} , X_{14} , and X_{15} of these nuclear species.
[Hint: The total number density of CN elements is conserved.]
- c) Estimate the time it takes for the establishment of equilibrium among the CN nuclei in the core of the Sun. Use your estimate to determine whether the CN nuclei already achieved their equilibrium values in the Sun's core. (Note: $\rho X \simeq 5 \times 10^4$ kg m⁻³ in the Sun's core.)
[Hint: What reaction determines the time required to achieve equilibrium?]
- d) In the Sun, $\sim 90\%$ of the nuclear energy is produced via the PPI chain, $\sim 9\%$ via the PPII and PPIII chains, and only $\sim 1\%$ via the CN cycle. Use this information to estimate the flux of PP neutrinos, ¹³N neutrinos and ¹⁵O neutrinos received at Earth. (For simplicity, assume that the available energy released per helium nucleus formed is 26 MeV for all energy generation channels).

8. Degenerate Matter.

Show that an electrically neutral gas, consisting of positive ions and fully relativistic and completely degenerate electrons, where the positive ions contribute negligibly to the pressure, obeys a polytropic equation of state of the form

$$P = K \rho^{(n+1)/n},$$

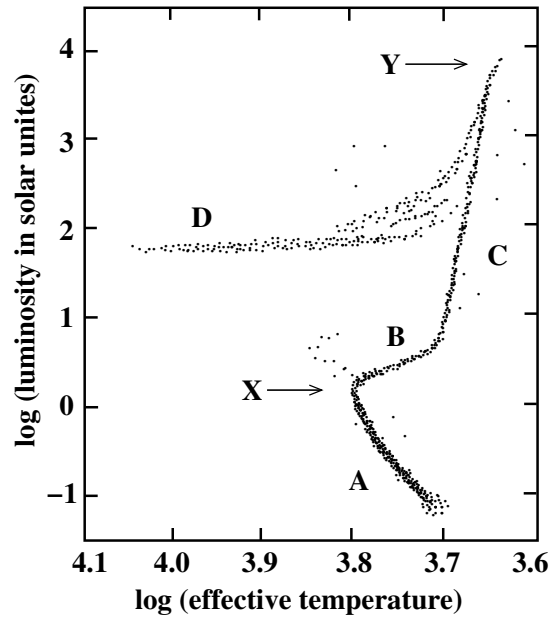
where ρ is the mass density. Show that the polytropic index $n = 3$ and derive the value of K .

The mass of a polytrope of index 3 is given approximately by

$$M = 8\pi \left(\frac{(n+1)K}{4\pi G} \right)^{3/2}.$$

Derive the value of the mass, M , of a white dwarf containing no hydrogen which obeys a fully-relativistic, completely-degenerate equation of state. Why is this an upper limit for a stable white dwarf? What is the likely final evolutionary state of stars more massive than this?

9. Hertzsprung-Russell Diagrams and Age Determinations.



The figure shows a schematic Hertzsprung-Russell (H-R) diagram typical of a globular cluster.

- Account for the distribution of stars shown in the figure. Explain, in particular, the significance of the turning points X and Y , and discuss the likely structure of stars in each of the regions A , B , C and D .
- Make a rough estimate of the age of the cluster given that $m_{\text{H}} = 1.0078$ a.m.u. and $m_{\text{He}} = 4.0026$ a.m.u.
- Sketch evolutionary tracks on the H-R diagram to illustrate the further evolution of stars in region D .

10. The Eddington Limit.

- Consider a fluid element at the surface of a star. Show that this element experiences an outward force due to the momentum deposited in it by radiation. By balancing this force with the gravitational force (or otherwise), show that there is a maximum luminosity for a star of mass M , known as the Eddington limit,

$$L_{\text{Edd}} = \frac{4\pi GMc}{\kappa}.$$

For stars like the Sun, the mass-luminosity relation is approximately given by $L \simeq L_{\odot} (M/M_{\odot})^4$. Argue how the existence of the Eddington limit implies a maximum

mass for stars in hydrostatic equilibrium. Discuss what happens when a star as a result of its internal evolution reaches this maximum luminosity?

- b) In X-ray binaries and active galactic nuclei (AGN), it is thought that most of the observed luminosity, L , is caused by the accretion of matter onto a compact object (a neutron star or black hole) and the conversion of gravitational potential energy into radiation (i.e. $L = (GM\dot{M})/R$, where M is the mass of the accreting object, R the inner radius of the accretion flow and \dot{M} is the mass accretion rate, $\dot{M} \equiv dM/dt$). Determine the maximum possible accretion rate in $M_{\odot} \text{ yr}^{-1}$ that does not violate the Eddington limit (assuming spherical accretion) for a neutron star (with $M = 1.4 M_{\odot}$ and $R = 10 \text{ km}$) and a supermassive black hole (with $M = 10^8 M_{\odot}$ and an inner accretion radius of $R = 9 \text{ km} (M/M_{\odot})$) [take κ to be $0.034 \text{ m}^2 \text{ kg}^{-1}$, the value for pure Thomson scattering in a solar-type plasma]. Discuss how the Eddington limit might be violated in nature.
- c) [*Harder:*] Assuming that a black hole continues to accrete at its Eddington limit, estimate the timescale on which the black-hole mass grows. How much can a black hole with an initial mass of $100 M_{\odot}$ grow in 10^9 yr ? Discuss this result in view of the fact that supermassive black holes as massive as $10^{10} M_{\odot}$ already seem to have existed when the age of the Universe was 10^9 yr . [Hint: you need to derive a differential equation for the mass of the black hole and solve it.]

11. Binary Break-up, Runaway Stars and Supernova Kicks.

Runaway stars are O and B stars which: (i) have large space velocities (up to 200 km/s , compared to 10 or 20 km/s for most stars of this spectral type); and (ii) are always single, whereas most O and B stars are in binary systems. One possible explanation is that they were formed as a member of a binary system, but that the more massive star (of mass m_1) reached the end of its life, exploded as a supernova and ejected most of its mass; as a result the less massive star (of mass m_2) became unbound, left the binary with its original orbital velocity and now appears as a runaway star.

Suppose that the two stars were initially in a circular orbit and that the more massive star instantaneously ejects a mass Δm in the supernova explosion. Assume that the ejection is spherically symmetric in the rest frame of the star. Show that the remnant becomes unbound after the supernova (i.e. show that the total energy of the post-supernova system is positive) if $\Delta m \geq (m_1 + m_2)/2$.

[*Harder:*] Observations of young neutron stars (pulsars) have recently shown that supernova explosions are not perfectly symmetric and that the neutron star receives a recoil velocity (“kick”) as a result of the supernova with a typical value of 250 km/s . Explain (e.g. using a suitable diagram or considering a special case) how, in this case, a binary system can remain bound even if $\Delta m \geq (m_1 + m_2)/2$ (you may assume that the recoil occurs in a random direction in the frame of the exploding star).