Stellar Evolution

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Homologous Stellar Models and Polytropes

Main Sequence Stars

Post-Main Sequence Hydrogen-Shell Burning

Post-Main Sequence Helium-Core Burning

White Dwarfs, Massive and Neutron Stars

This series of ten lectures introduces stellar astrophysics and discusses stellar evolution.

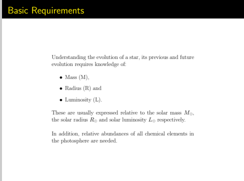
Basic Requirements

Understanding the evolution of a star, its previous and future evolution requires knowledge of:

- Mass (M),
- Radius (R) and
- Luminosity (L).

These are usually expressed relative to the solar mass $M_\odot$, the solar radius $R_\odot$ and solar luminosity $L_\odot$ respectively.

In addition, relative abundances of all chemical elements in the photosphere are needed.



While mass, radius, luminosity and photospheric abundances characterise a star's evolutionary state, the mass-loss rate is another critical parameter in the case of massive, binary and giant stars. The neutrino flux where it can be observed, as in the case of the Sun, gives vital information about conditions in the stellar core. Further insight into the internal structure may be obtained from the study of pulsations in cases where a star pulsates.

This first lecture will give a brief overview of stellar mass, radius, and luminosity determinations. The second lecture will give an equally brief summary of how abundances are determined.

Effective Temperature and Angular Radius

$$\sigma {T_{\mathrm{eff}}}^4 = \int_0^\infty F_\lambda \, d\lambda$$

$$L = 4\pi R^2 \int_0^\infty F_\lambda \, d\lambda = 4\pi d^2 \int_0^\infty f_\lambda \, d\lambda$$

$$\alpha^2 = \frac{R^2}{d^2} = \frac{f_\lambda}{F_\lambda}$$

$$L = 4\pi R^2 \sigma {T_{\mathrm{eff}}}^4$$

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Define F_λ to be a monochromatic flux per unit area, per unit wavelength interval emergent from the stellar surface; integrating this quantity over all wavelengths gives an integrated flux per unit area which is identical to that of a black body whose temperature is T_{eff} as specified in the first equation, where σ is Stefan's constant. The quantity T_{eff} is defined to be the stellar effective temperature.

If d is the stellar distance and f_λ the corresponding monochromatic flux observed at the top of the Earth's atmosphere then the second equation gives L (energy generated by the star per second) in terms of R or d. The third equation then gives the squared stellar angular radius α^2. If α can be measured by interferometry or lunar occultation, F_λ follows once f_λ has been observed. Alternatively, F_λ may be predicted using a model of the stellar atmosphere and α then follows. A correction for interstellar reddening is usually needed.

Effective Temperature and Surface Gravity Scaling

$$\frac{L}{L_\odot} = \left(\frac{R}{R_\odot}\right)^2 \left[\frac{T_{\rm eff}}{(T_{\rm eff})_\odot}\right]^4$$

$$\frac{g}{g_\odot} = \frac{M}{M_\odot}\left(\frac{R}{R_\odot}\right)^{-2}$$

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Surface gravity (g) and $T_{\rm eff}$ may be determined from observations and these scaling relations relating them to L, M and R are therefore extremely important. $T_{\rm eff}$ is the most important parameter which is immediately apparent from an inspection of stellar energy distributions or spectra; it determines (roughly) the wavelength at which the maximum flux density occurs and the slope of the energy distribution at any arbitrary wavelength. In addition, $T_{\rm eff}$ is related to the ionisation fraction, the fraction of each species associated into molecules and level populations. Pressure in stellar atmospheres is closely related to g and the density of perturbers (electrons and neutral hydrogen atoms being the most important) which produce line broadening.

Kepler's Third Law

van den Bos WH, 1961 MNASSA 20 138

Planet	a	P	a^3	P^2
Mercury	0.388	0.241	0.0584	0.0581
Venus	0.724	0.615	0.3795	0.3785
Earth	1	1	1	1
Mars	1.5235	1.881	3.536	3.538
Jupiter	5.1965	11.857	140.3	140.6
Saturn	9.510	29.425	860.1	865.8

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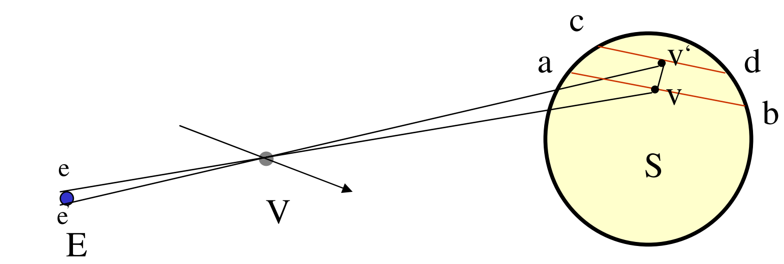
As may be noted from an earlier slide, stellar luminosity can only be derived from a monochromatic flux density measured at the top of the Earth's atmosphere if the stellar distance is known. Crucial to the determination of stellar distances is a knowledge of the scale of the Solar System. Relative distances of planets from the Sun may be deduced from Kepler's Third Law.

Take barycentric distances of various planets (a) relative to that for the Earth ($a = 1$) and compute a^3. Expressing the orbital periods P of the same planets in years (*i.e.* $P = 1$ for the Earth) and compare P^2 with a^3; the two are found to be almost equal, which is Kepler's Third Law. Thus by observing orbital periods of the planets we can fix relative distances in the Solar System but not its absolute scale.

Earth-Sun Distance Measurement

Metz D, 2009 Science & Education 18 581



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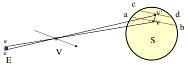
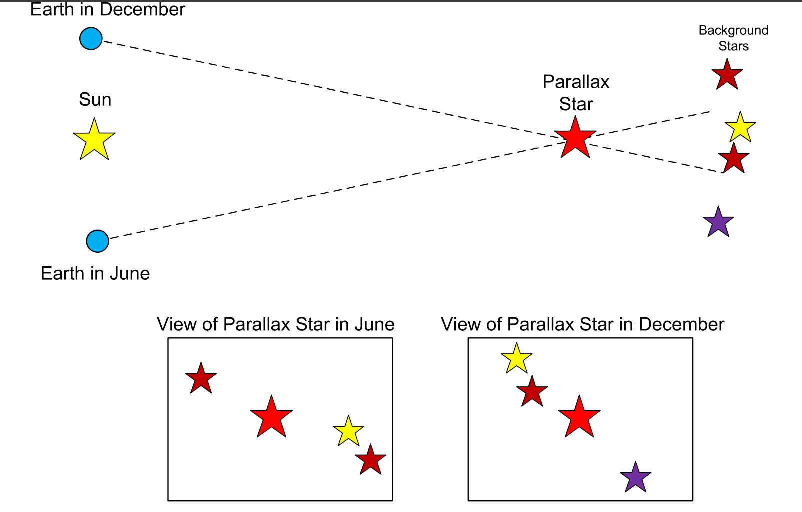
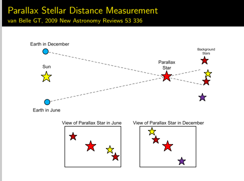


Halley's method for determining the Earth-Sun distance, or Astronomical Unit (AU), relies on two observers on the Earth's surface (at e and $e^{'}$ who observe Venus to transit across the solar disk along trajectories ab and cd respectively. From Kepler's Third Law, $EV/VS = 7/18$ and since the triangles are similar $VV^{'} = 18 \times ee^{'}/7$. Knowing $VV^{'}$ gives the Sun's radius ($R_{\odot}$) and therefore the AU since the angle subtended by the Sun's disk, as seen by an observer on Earth, has already been measured. However, the Earth and Venus both orbit the Sun and the Earth and Sun both rotate. Moreover, it was unlikely that two eighteenth century observers at e and $e^{'}$ could make simultaneous observations. Halley therefore proposed that the trajectories ab and cd both be timed so that they could be subsequently reconstructed to allow a $VV^{'}$ determination.

Parallax Stellar Distance Measurement

van Belle GT, 2009 New Astronomy Reviews 53 336

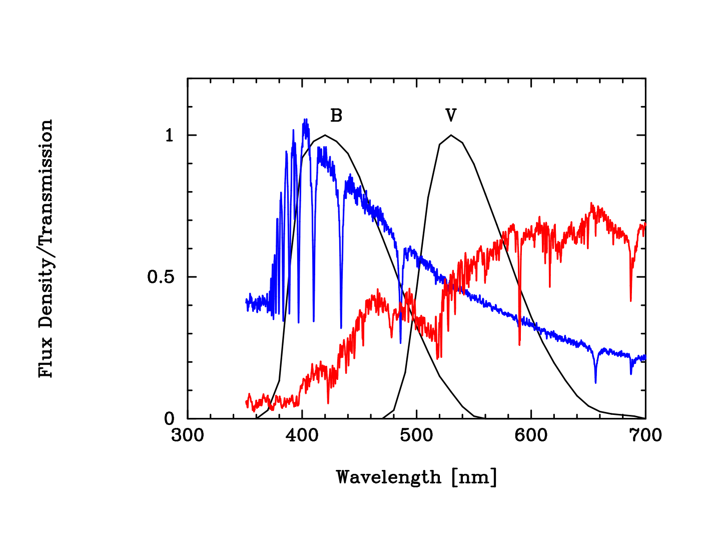


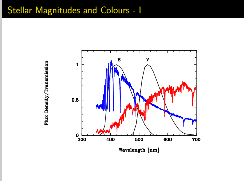


The apparent position of a nearby star changes slightly, with respect to more distant background stars, when observed six months later. The position change relative to the background stars gives the angle subtended by the Earth's orbit diameter at the nearby star. Knowing the diameter of the Earth's orbit then yields the distance of the nearby star. Of course, proper motion complicates the measurement and needs to be taken into account by repeating the observations over several years.

A hypothetical star at which the diameter of the Earth's orbit subtends an angle of 1 arcsecond is said to be at a distance of 1 parsec (pc). From the ground it is possible to measure distances to 50 pc by parallax; the Hipparcos mission extended this to 500 pc. Parallaxes for many more distant stars are anticipated over the next few years from the GAIA satellite launched at the end of 2013.

Stellar Magnitudes and Colours - I





Stellar magnitudes ("brightness") and colours provide additional information as the former is related to the luminosity and the latter to temperature. The diagram shows scaled energy distributions for HD 116608 (a hot star – blue line) and BD $+63^{\circ}00137$ (a cool star – red line). It is customary to collect light from stars through filters; the transmission functions for two such filters (labelled "B" and "V") are superimposed on the stellar energy distributions in the diagram. Clearly the hot star will give rise to a stronger signal in the B-filter than in the V-filter; for the star, the reverse is the case. As explained in the next slide, the quantity $(B - V)$ is expressed in magnitudes and characterises the stellar colour (and therefore the stellar temperature)

Stellar Magnitudes and Colours - II

$$m_2 - m_1 = -2.5 \log_{10}(F_2/F_1)$$

$$V = V_0 - 2.5 \log_{10} \left[\frac{\displaystyle\int_0^\infty F_\lambda S_\lambda(V)\, d\lambda}{\displaystyle\int_0^\infty S_\lambda(V)\, d\lambda} \right]$$

$$\lambda_{\mathrm{eff}} = \left[\frac{\displaystyle\int_0^\infty \lambda F_\lambda S_\lambda(V)\, d\lambda}{\displaystyle\int_0^\infty F_\lambda S_\lambda(V)\, d\lambda} \right]$$

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The first equation gives the magnitude difference $(m_2 - m_1)$ which corresponds to two observed fluxes or flux densities, the later being fluxes per unit wavelength interval, F_1 and F_2; the numerically larger magnitude corresponds to the lower flux or flux density for historical reasons. F_1 and F_2 may arise from the same star observed through different filters in which case it is a colour measurment for that star; for different stars observed through the same filter, it is a differential magnitude.

If $S_\lambda(V)$ is the wavelength-dependent transmission of the Johnson V-Band filter corrected for wavelength dependencies in the detector, transmission of the Earth's atmosphere and interstellar medium, then the second equation defines the Johnson V-Band magnitude. Here V_0 is a constant chosen so $V = 0$ when F_λ is the flux density of Vega as observed at the top of the Earth's atmosphere. In the second equation

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the flux density is convolved with the effective filter transmission and therefore needs to be normalised through division by the "area" of that effective filter transmission function. Replacing $S_\lambda(V)$ by $S_\lambda(B)$, V by B and V_0 by B_0 gives the corresponding Johnson B-Magnitude. Forming the difference $(B - V)$ gives commonly used stellar colour index.

Filter observations are sometimes used to measure a monochromatic stellar flux. The effective wavelength (λ_{eff}) is a mean wavelength across the filter, weighted by the wavelength-dependent stellar flux density and effective filter transmission functions, as given by the third equation.

Absolute Magnitude

$$f_\lambda(V) = \left(\frac{\mathcal{D}}{d}\right)^2 \mathcal{F}_\lambda(V)$$

$$\begin{aligned} m_V - \mathcal{M}_V &= 2.5 \log_{10}\left(\frac{\mathcal{F}_\lambda(V)}{f_\lambda(V)}\right) \\ &= 2.5 \log_{10}\left(\frac{d}{\mathcal{D}}\right)^2 \\ &= 5 \log_{10} d - 5 \end{aligned}$$

(when $\mathcal{D} = 10\text{pc}$).

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We have seen how the monochromatic flux per unit area (or integrated over a surface) scales with the inverse square of the distance. As stars have different luminosities, it is not only different distances that contribute to observed magnitudes. A standard distance, selected to be 10 pc, at which to compare magnitudes is needed; the magnitude that a star would have if it were at this distance is the absolute magnitude. Equations presented in the slide show how a relation is obtained for relating absolute magnitude $\mathcal{M}_V$ to the observed magnitude m_V and distance d in pc.

Bolometric Correction

$$\ell = \int_0^\infty f_\lambda \, d\lambda$$

$$m_{\mathrm{bol}} - m_V = 2.5 \log_{10} \frac{\ell(V)}{\ell}$$

$$= 2.5 \log_{10} \left[\frac{\displaystyle\int_0^\infty f_\lambda(V) S_\lambda(V) d\lambda}{\displaystyle\int_0^\infty f_\lambda \, d\lambda} \right]$$

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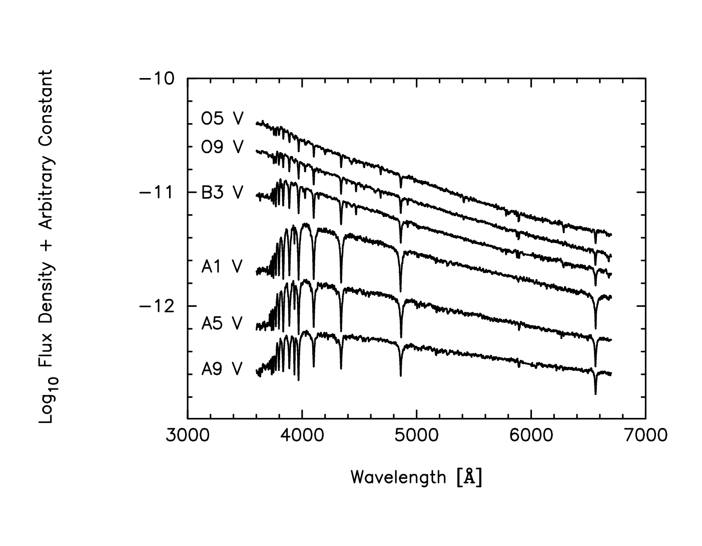
In general, only a fraction of the stellar flux is emitted at wavelengths to which the human eye is sensitive or which are encompassed within the V-filter pass-band. It is therefore standard practise to use what is termed a **Bolometric Correction** which when added to m_V (the V-filter magnitude), gives a bolometric magnitude ($m_{\rm bol}$) which is the magnitude the star would have if all flux were included in the magnitude calculation. For the Sun, most flux emerges in the V-band and the Bolometric Correction is small. Most flux from hot stars emerges in the ultraviolet and bolometric corrections can be several magnitudes. Similarly for cool stars where most flux emerges in the infrared.

Stellar Type - Temperature Classification - I



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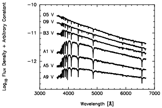
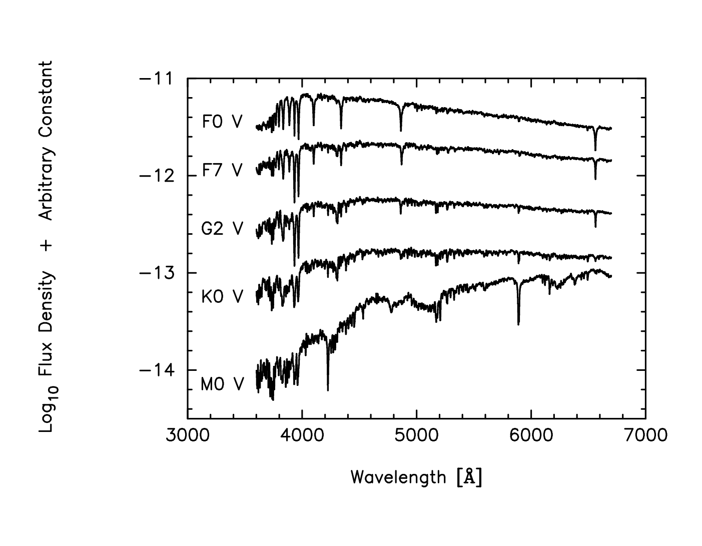
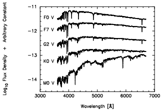


This slide compares energy distributions of the hotter stars evolving on the Main Sequence. Note the increasing proportion of flux emerging in the ultraviolet as $T_{\rm eff}$ increases from A9 V to O5 V. Also see how the strength of the Balmer lines is a maximum at A1 V; hydrogen becomes increasingly ionised at higher $T_{\rm eff}$ while at lower $T_{\rm eff}$ populating the upper levels of the hydrogen atom becomes increasingly less probable. Helium also becomes increasingly ionised as $T_{\rm eff}$ increases but has a higher ionisation potential. He I lines have maximum strength at B2 while He II lines are strongest at O5 V.

Stellar Type - Temperature Classification - II

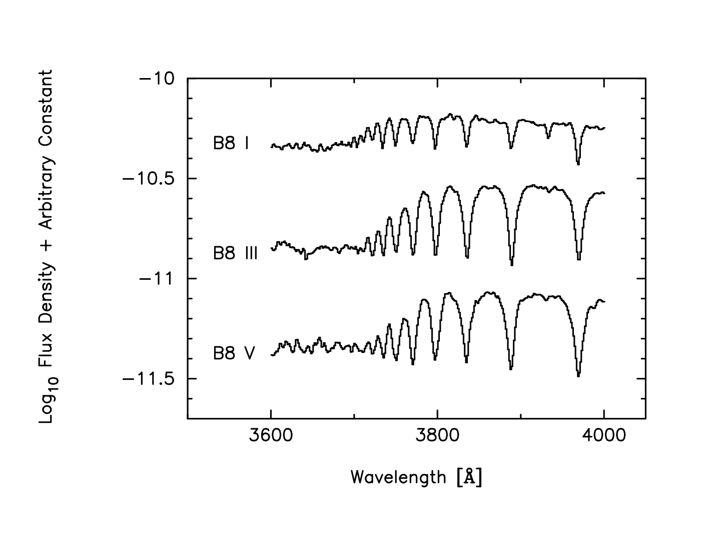


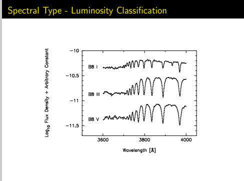
Stellar Type - Temperature Classification - II



Here the energy distributions of the cooler stars evolving on the Main Sequence are compared. Balmer lines become increasingly weak as $T_{\rm eff}$ is reduced and eventually disappear. Metal lines begin to appear because, with hydrogen neutral, there are essentially no free electrons and therefore no electron scattering opacity. Moreover, metals have a lower ionisation potential than hydrogen and become increasingly neutral as $T_{\rm eff}$ falls. Neutral metal and hydrogen atoms become increasingly associated into molecules in the coolest stars and eventually absorption bands due to polyatomic molecules dominate the stellar spectra.

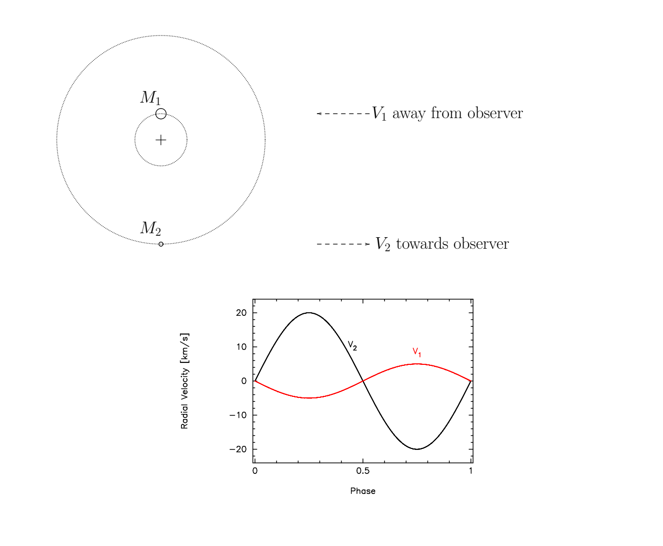
Spectral Type - Luminosity Classification

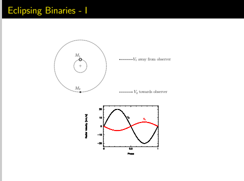




The slide shows how spectra change with decreasing surface gravity at a fixed $T_{\rm eff}$. Not only does the width of the lines decrease as atmospheric pressure decreases from $\rm B8\ V \rightarrow B8\ I$ but the number of Balmer lines that may be counted on the long wavelength side of the Balmer jump increases. A lower atmospheric pressure leads to fewer and more distant electron collisions with radiating atoms and hence reduced pressure broadening. Another consequence of reduced atmospheric pressure is that higher lying levels remain bound and hence Balmer lines corresponding to electron jumps from the $n = 2$ level to these higher lying levels are seen in the spectra.

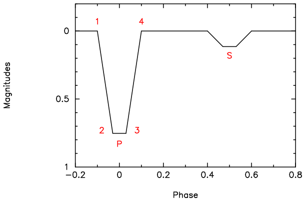
Eclipsing Binaries - I

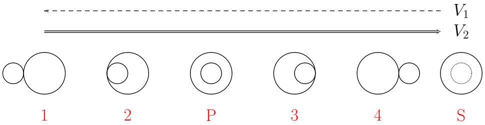


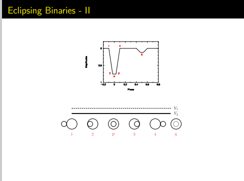


Stars in a binary system whose orbital plane lies in the line-of-sight will eclipse each other. Supposing the simplest case of circular orbits, the centripetal force is provided by the gravitational attraction. Radial velocity curves of both stars will be sinusoidal and in anti-phase; these can be measured if the magnitudes of the two stars is not too different. Knowing the period and orbital speeds which can be measured from the radial velocity curves (once corrected for a systemic velocity), the radii of the orbits and stellar masses can be obtained.

Eclipsing Binaries - II

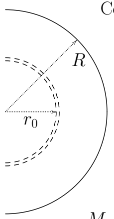






A schematic light curve is shown where the "contact points" 1, 2, 3 and 4 are marked along with the primary (P) and secondary (S) eclipses. Radii of the two stars follow from a knowledge of their orbital speeds and the times taken between first and fourth contacts, and between second and third contacts. In both cases T_{eff} may be determined with the usual spectroscopic methods. Luminosities for both stars in the binary would then follow. As can be seen, eclipsing binary stars allow a confident determination of fundamental stellar parameters (mass, radius and luminosity) for various spectral types. A spectrum of a field star can then give a luminosity, which in turn provides a distance and so enables the distance scale to be extended to well beyond what may be achieved with the parallax method.

Radial Pulsation - I



Consider spherical shell of thickness dr_0 with equilbrium distance r_0 from centre of star of radius R.

P_0 - equilibrium pressure at r_0.

ρ_0 - equilibrium density at r_0.

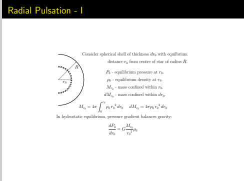
M_{r_0} - mass confined within r_0.

dM_{r_0} - mass confined within dr_0.

$$M_{r_0} = 4\pi \int_0^{r_0} \rho_0 \, {r_0}^2 \, dr_0 \qquad dM_{r_0} = 4\pi \rho_0 \, {r_0}^2 \, dr_0$$

In hydrostatic equilibrium, pressure gradient balances gravity:

$$\frac{dP_0}{dr_0} = G \frac{M_{r_0}}{{r_0}^2} \rho_0$$



The hydrostatic equilibrium equation is introduced along with the framework in which radial pulsations are discussed in the context of small perturbations to radial shells.

Radial Pulsation - II

Compress star and release; it then oscillates (or pulsates) leading to a time-dependent radial shift of mass shells within the star:

$$\frac{\Delta r}{r_0} = x(t) \quad \text{or} \quad r(t) = r_0\left[1 + x(t)\right] \quad \text{and} \quad dr = \left[1 + x(t)\right] dr_0$$

where $x(t)$ is a time-dependent perturbation. If the shells conserve their mass ($M_r = M_{r_0}$)

$$\rho r^2 dr = \rho_0 {r_0}^2 dr_0,$$

and do not exchange energy, the pulsation is adiabatic:

$$P = P_0 \left(\frac{\rho}{\rho_0}\right)^{\gamma}$$

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The time-dependent perturbation in radius is introduced as a consequence of "compressing a star and releasing it". The perturbation introduced ($x(t)$) is a fractional radius change in units of the equilbrium shell radius (r_0). If concentric spherical shells which make up the star move up and down together, no mass will be exchanged between them. Each shell conserves its mass and if energy is not exchanged between shells, the pulsation will be adiabatic and the adiabatic gas pressure – density relation may be adopted.

Radial Pulsation - III

Then for $x(t) \ll 1$

$$\rho = \rho_0 \left[1 + x(t)\right]^{-3} \simeq \rho_0 \left[1 - 3x(t)\right],$$

$$P = P_0 \left[1 + x(t)\right]^{-3\gamma} \simeq P_0 \left[1 - 3\gamma x(t)\right]$$

$$\text{and} \quad \frac{1}{r^2} \simeq \frac{1}{{r_0}^2} \left[1 - 2x(t)\right].$$

Hydrostatic equilibrium no longer applies; shell acceleration needs to be included in the equation of motion (EOM):

$$\frac{dP}{dr} = -G \frac{M_r}{r^2} \rho - \rho \frac{d^2 r}{dt^2}$$

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Perturbations in terms of equilibrium values then follow for density, pressure and inverse squared shell radius. Because $x(t) \ll 1$, the Binomial Theorem may be applied and terms in $x(t)^2$ and higher orders may be neglected, leaving simple linear relations in each case. Once the star has been perturbed out of its equilibrium structure, hydrostatic equilibrium no longer exists. The Equation of Hydrostatic Equilibrium needs replacing with an equation of motion in which the pressure gradient is balanced by the gravitational restoring force on the shell and the force causing its acceleration.

Radial Pulsation - IV

Left Hand Side of EOM:

$$\frac{dP}{dr} = \frac{dP}{dr_0}\frac{dr_0}{dr} = \frac{dP_0}{dr_0}\frac{1-3\gamma x}{1+x}$$

$$\simeq \frac{dP_0}{dr_0}(1-3\gamma x)(1-x) = G\frac{M_{r_0}}{{r_0}^2}\rho_0\,(1-3\gamma x)(1-x)$$

$$\simeq G\frac{M_{r_0}}{{r_0}^2}\rho_0\,(1-x(3\gamma+1))$$

Right Hand Side of EOM:

$$G\frac{M_r}{r^2}\rho = G\frac{M_{r_0}}{{r_0}^2}(1-2x)\rho_0(1-3x) \simeq G\frac{M_{r_0}}{{r_0}^2}\rho_0(1-5x)$$

and

$$-\rho\frac{d^2r}{dt^2} = -\rho_0\,(1-3x)\,r_0\,\frac{d^2x}{dt^2}$$

Radial Pulsation - IV

Left Hand Side of EOM:

$$\frac{dP}{dr} = \frac{dP}{dr_0}\frac{dr_0}{dr} = \frac{dP_0}{dr_0}\frac{1-3\gamma x}{1+x}$$
$$\simeq \frac{dP_0}{dr_0}(1-3\gamma x)(1-x) = G\frac{M_{r_0}}{{r_0}^2}\rho_0(1-3\gamma x)(1-x)$$
$$\simeq G\frac{M_{r_0}}{{r_0}^2}\rho_0(1-x(3\gamma+1))$$

Right Hand Side of EOM:

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and

$$-\rho\frac{d^2r}{dt^2} = -\rho_0(1-3x)r_0\frac{d^2x}{dt^2}$$

The time-dependent and perturbed pressure gradient on the left-hand side of the equation of motion is expressed in terms of the equilibrium pressure gradient and shell displacement ($x(t)$) in units of the equilibrium shell radius. The same is done for the gravitational restoring on the shell which is also expressed in terms of the equilibrium values and shell displacement in units of the equilibrium shell radius. The second term on the right-hand side of the equation of motion is the force accelerating the shell; this is expressed in terms of the equilibrium shell density, the equilibrium shell radius, $x(t)$ and the second derivative of $x(t)$ with respect to time.

Radial Pulsation - V

EOM becomes:

$$G\,\frac{M_{r_0}}{{r_0}^2}\,\rho_0\,(1 - x(3\gamma + 1)) = G\,\frac{M_{r_0}}{{r_0}^2}\rho_0(1 - 5x) - \rho_0\,(1 - 3x)\,r_0\,\frac{d^2x}{dt^2}$$

$$r_0\,\frac{d^2x}{dt^2} = G\,\frac{M_{r_0}}{{r_0}^2}\,\frac{(4 - 3\gamma)}{1 - 3x}\,x$$

The boundary condition is that $r = r_0(1 - x(t)) = R$ and $M_{r_0} = M$ at the stellar surface giving:

$$\frac{d^2x}{dt^2} + (3\gamma - 4)\,\frac{3G\bar{\rho}}{4\pi}x = 0, \quad \text{with} \quad \bar{\rho} = \frac{M}{(4\pi/3)\,R^3}$$

For pulsation period Π, the solution is:

$$x = x_0\,\exp(i\omega t) \quad \text{with} \quad \omega^2 = 4\pi^2/\Pi^2 = (3\gamma - 4)\,\frac{3G\bar{\rho}}{4\pi}.$$

Radial Pulsation - V

EOM becomes:

$$G\frac{M_{r_0}}{r_0^2}\rho_0(1-x(3\gamma+1)) = G\frac{M_{r_0}}{r_0^2}\rho_0(1-5x) - \rho_0(1-3x)r_0\frac{d^2x}{dt^2}$$

$$r_0\frac{d^2x}{dt^2} = G\frac{M_{r_0}}{r_0^2}\frac{(4-3\gamma)}{1-3x}x$$

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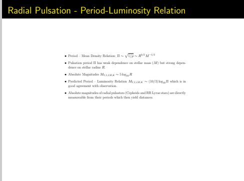
On simplfying the resulting equation of motion, we end up with a second-order differential equation in $x(t)$ which has no first-order term; this is the equation of simple harmonic motion which has the well-known sinusoidal motion as a solution. The period of oscillation is proportional to the square root of the mean density, which is the period mean density relationship.

Radial Pulsation - Period-Luminosity Relation

- Period – Mean Density Relation: $\Pi \sim \sqrt{1/\bar{\rho}} \sim R^{3/2}\,M^{-1/2}$
- Pulsation period Π has weak dependence on stellar mass (M) but strong dependence on stellar radius R.
- Absolute Magnitudes $M_{V,I,J,H,K} \sim 5\log_{10} R$
- Predicted Period – Luminosity Relation $M_{V,I,J,H,K} \sim (10/3)\log_{10}\Pi$ which is in good agreement with observation.
- Absolute magnitudes of radial pulsators (Cepheids and RR Lyrae stars) are directly measureable from their periods which then yield distances.



Ignoring the weak dependence of the radial pulsation period on stellar mass, it follows that $\log_{10} \Pi \sim 3/2 \log_{10} R$. Moreover $M_{\mathrm{V}} \sim 2.5 \log_{10} L \sim 5 \log_{10} R$ and so $M_{\mathrm{V}} \sim 10/3 \log_{10} \Pi$.

Stellar Masses, Radii & Luminosities

Spectrum	$\log_{10}(M/M_\odot)$			$\log_{10}(R/R_\odot)$			$\log_{10}(L/L_\odot)$		
	I	III	V	I	III	V	I	III	V
B0	+1.70		+1.23	+1.30	+1.20	+0.88	+5.50		+4.10
A0	+1.20		+0.55	+1.60	+0.80	+0.42	+4.40		+1.90
F0	+1.10		+0.25	+1.80		+0.13	+3.90		+0.80
G0	+1.00	+0.40	+0.03	+2.00	+0.80	+0.02	+3.80	+1.50	+0.10
K0	+1.10	+0.60	−0.09	+2.30	+1.20	−0.07	+4.00	+2.00	−0.40
M0	+1.20	+0.80	−0.32	+2.70		−0.20	+4.50	+2.60	−1.20

Stellar Masses, Radii & Luminosities

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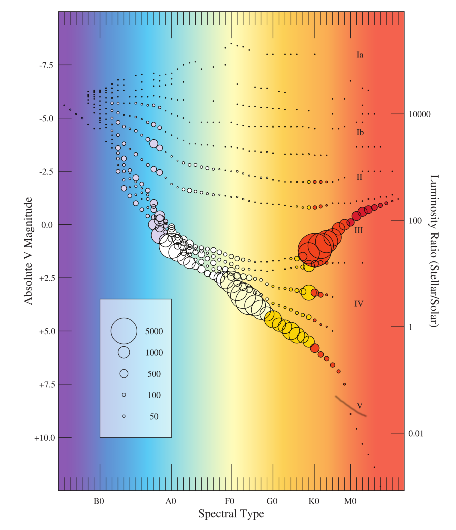
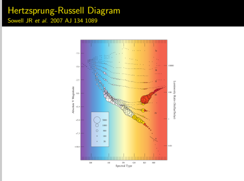
M0	+1.20	+0.80	−0.32	+2.70		−0.20	+4.50	+2.60	−1.20

Stellar masses, radii and luminosities presented in the table are approximate and should not be deemed to be a calibration against spectral type. The idea is to illustrate changes in mass, radius and luminosity between luminosity class V (Main Sequence), luminosity class III (Giant) and luminosity class I (Supergiant) within a spectral class. Also worthy of note is the variation in mass, radius and luminosity within (for example) luminosity class V.

Hertzsprung-Russell Diagram

Sowell JR *et al.* 2007 AJ 134 1089

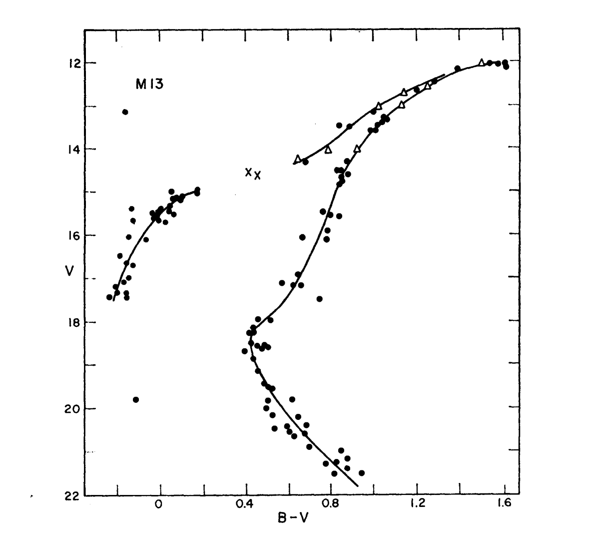




Explain distribution of stars on the Main Sequence and Red Giant Branches. Argue that stars therefore spend most of their lives on the Main Sequence and models of stellar evolution must explain this. Point out the locations of hot subdwarfs and white dwarfs, indicating that there are not many of these and that they are therefore short-lived stages stellar evolution.

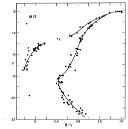
Colour-Magnitude Diagram for the Globular Cluster M13

Sandage A 1970 ApJ 162 841



Colour-Magnitude Diagram for the Globular Cluster M13

Sandage A 1970 ApJ 162 841



A colour-magnitude diagram for M3 where Sandage's photoelectric photometry is shown as large filled circles. Open triangles are points from an earlier photographic study, Crosses are possible blue stragglers and small dots are from a study of the Main Sequence by Katem and Sandage. Note that all stars are understood to be of the same age and so the Main Sequence turnoff colour (from which a mass may be inferred from evolution models) is an age indicator.

Lecture 1: Summary

Essential points covered in first lecture:

- Distances of nearby stars may be measured by parallax once scale of the Solar System has been established.
- Stellar radii and masses may be determined through the study of eclipsing binary stars.
- Stars may be classfied spectroscopically; those with the same spectra have the same masses, radii and luminosities and this may be used to extend the distance scale.
- Radially pulsating stars such as Cepheids and RR Lyraes serve as distance indicators, once their pulsation periods are known, through the period-luminosity-relation.
- Distributions of stars in colour-magnitude diagrams needs to be explained by stellar evolution theory and models.

Stellar evolution depends on initial photospheric abundances and their determination from spectra are to be discussed in the next two lectures.

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While mass, radius, luminosity and photospheric abundances characterise a star's evolutionary state, the mass-loss rate is another critical parameter in the case of massive, binary and giant stars. The neutrino flux where it can be observed, as in the case of the Sun, gives vital information about conditions in the stellar core. Further insight into the internal structure may be obtained from the study of pulsations in cases where a star pulsates.

This first lecture gives a brief overview of stellar mass, radius, and luminosity determinations. The second and third lectures will give an equallycbrief summary of how abundances are determined.

Acknowledgement

Material presented in this lecture on radial pulsation and the period-luminosity relation is based on slides prepared by R.-P. Kudritzki (University of Hawaii).