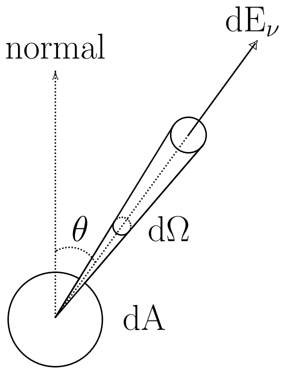
Radiative Transfer Introduction

Relative abundances of chemical elements are adjusted until synthetic spectra agree with observation; requirements are:

- Accurate laboratory determinations of atomic and molecular data, primarily oscillator strengths.
- $T_{\rm eff}$ and $\log g$ from some combination of energy distributions, Balmer line profiles, ionisation ratios and molecular association ratios.
- Temperature, gas pressure and electron pressure dependence on geometric or optical depth in the stellar atmosphere.
- Broadening theory for lines used in abundance determinations.
- Radiation transfer theory to calculate the wavelength-dependent emergent flux

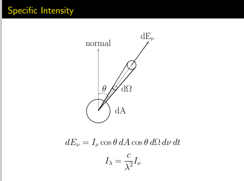
Dependent on radiative transfer; the subject of this lecture.

Specific Intensity



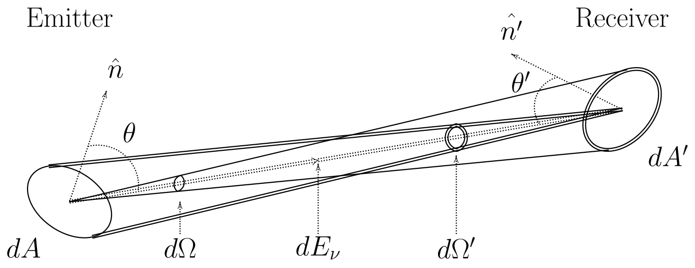
$$dE_\nu = I_\nu \cos\theta \, dA \cos\theta \, d\Omega \, d\nu \, dt$$

$$I_\lambda = \frac{c}{\lambda^2} I_\nu$$



Consider an infinitesimal area dA which radiates energy dE_ν in time dt, in frequency interval $d\nu$, into solid angle $d\Omega$ and in a direction inclined at an angle θ to the normal to dA. The projected infinitesmal area in the direction of dE_ν is $dA \cos\theta$. Clearly dE_ν is proportional to dt, $d\nu$, $d\Omega$ and $dA \cos\theta$. $I_\nu \cos\theta$ is adopted as the "constant" of proportionality, the $\cos\theta$ dependence arises because $dE_\nu = 0$ when $\theta = \pi/2$ and attains a maximum when $\theta = 0$.

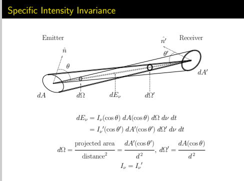
Specific Intensity Invariance



$$dE_\nu = I_\nu(\cos\theta)\; dA(\cos\theta)\; d\Omega\; d\nu\; dt$$
$$= I_\nu{}'(\cos\theta')\; dA'(\cos\theta')\; d\Omega'\; d\nu\; dt$$

$$d\Omega = \frac{\text{projected area}}{\text{distance}^2} = \frac{dA'(\cos\theta')}{d^2},\; d\Omega' = \frac{dA(\cos\theta)}{d^2}$$

$$I_\nu = I_\nu{}'$$



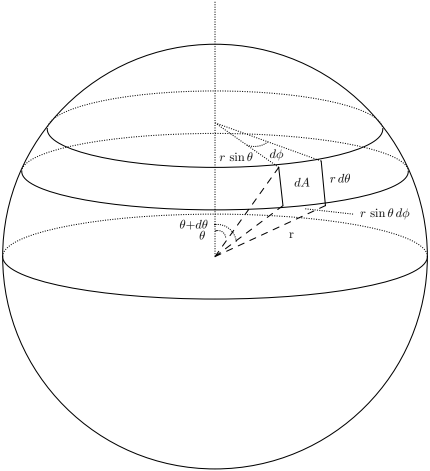
An infinitesimal area $dA^{'}$ acts as a receiver of an infinitesimal amount of energy dE_ν emitted by another infinitesimal area dA, at which $dA^{'}$ subtends an infinitesimal solid angle $d\Omega$. Similarly, dA subtends an infinitesimal solid angle $d\Omega^{'}$ at $dA^{'}$. As energy is conserved in the absence of any absorption or scattering along the line of sight, the dE_ν emitted by dA into $d\Omega$ is also received by $dA^{'}$ from $d\Omega^{'}$. After substituting for $d\Omega$ and $d\Omega^{'}$ in terms of the projected area divided by the squared distance between dA and $dA^{'}$, it follows that the specific intensity ${I_\nu}^{'}$ at the receiver is the same as the specific intensity I_ν emitted by the emitter.

Spherical Coordinates

$$d\Omega = \frac{dA}{r^2}$$

$$dA = (r\,d\theta)(r \sin\theta\,d\phi)$$

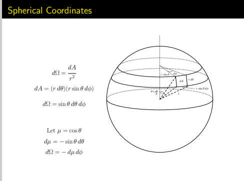
$$d\Omega = \sin\theta\,d\theta\,d\phi$$



Let $\mu = \cos\theta$

$$d\mu = -\sin\theta\,d\theta$$

$$d\Omega = -\,d\mu\,d\phi$$



Since it is usual to consider an infinitesimal area dA to reside on the surface of a sphere, it is customary to express it and an infinitesimal solid angle $d\Omega$ in terms of spherical polar coordinates (r, θ, ϕ). Here r is the radius of a sphere, θ the altitudinal angle and ϕ the azimuthal angle.

Radiative Flux

Energy flowing through element of area dA in unit time, per unit frequency interval, per unit area is the monochromatic physical flux F_ν:

$$
\begin{aligned}
F_\nu &= \pi\, {F_\nu}^a \\
&= \oint I_\nu(\cos\theta)\cos\theta\, d\Omega \\
&= \int_0^{2\pi}\int_0^{\pi} I_\nu(\cos\theta)\cos\theta\ \sin\theta\, d\theta\, d\phi \\
&= \int_0^{2\pi}\int_0^{\pi/2} I_\nu(\cos\theta)\cos\theta\ \sin\theta\, d\theta\, d\phi + \int_0^{2\pi}\int_{\pi/2}^{\pi} I_\nu(\cos\theta)\cos\theta\ \sin\theta\, d\theta\, d\phi \\
&= {F_\nu}^+ + {F_\nu}^-
\end{aligned}
$$

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For historical reasons, a monochromatic **astrophysical flux** ${F_\nu}^{\mathrm{a}}$ is defined such that when this quantity is multiplied by π, the monochromatic physical flux is obtained. The latter is defined as the energy flowing through an element of area dA at frequency ν in unit time, per unit frequency interval, per unit area; this is the specific intensity, with projection effects taken into account, integrated over all solid angles. Expressing the $d\Omega$ in terms of $d\theta$ and $d\phi$ leads to a double integral over θ and ϕ. Since $0 \leqslant \phi \leqslant 2\pi$ it follows that $0 \leqslant \theta \leqslant \pi$. The interval $0 \leqslant \theta \leqslant \pi/2$ corresponds to an emergent monochromatic flux (${F_\nu}^{+}$) whereas the interval $\pi/2 \leqslant \theta \leqslant \pi$ corresponds to an incident monochromatic flux (${F_\nu}^{-}$).

Stellar Luminosity

Energy flowing through element of area dA in unit time, per unit area is the total physical flux $F_{\rm rad}$:

$$F_{\rm rad} = \int_0^\infty F_\nu \, d\nu$$

In general, no incident flux at outer boundary of atmosphere ($r = R_0$):

$$\begin{aligned} F_{\rm rad} &= \int_0^\infty {F_\nu}^+ \, d\nu \\ &= \int_0^\infty \int_0^{2\pi} \int_0^{\pi/2} I_\nu(\cos\theta) \cos\theta \sin\theta \, d\theta \, d\phi \, d\nu \end{aligned}$$

Stellar luminosity is energy per unit time from the entire stellar surface:

$$L = 4\pi \, {R_0}^2 \, F_{\rm rad}$$

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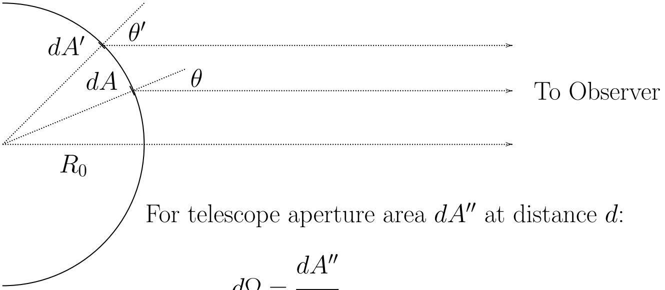
The total physical flux passing through an element of area is the integral over all frequencies of the monochromatic flux; this gives the energy per unit time and per unit area emerging from the stellar surface, in the usual case where there is no incident flux. Multiplication by the surface area gives the stellar luminosity.

Observed Flux - I

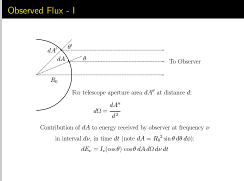


For telescope aperture area dA'' at distance d:

$$d\Omega = \frac{dA''}{d^2}$$

Contribution of dA to energy received by observer at frequency ν in interval $d\nu$, in time dt (note $dA = {R_0}^2 \sin\theta \, d\theta \, d\phi$):

$$dE_\nu = I_\nu(\cos\theta) \, \cos\theta \, dA \, d\Omega \, d\nu \, dt$$



The aperture of an observer's telescope, and distance from the star, determine the solid angle subtended at the observer's telescope by any element of area on the stellar surface.

Observed Flux - II

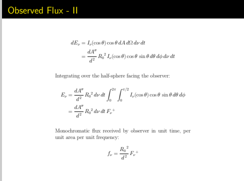
$$dE_\nu = I_\nu(\cos\theta)\cos\theta \, dA \, d\Omega \, d\nu \, dt$$
$$= \frac{dA''}{d^2} {R_0}^2 \, I_\nu(\cos\theta)\cos\theta \, \sin\theta \, d\theta \, d\phi \, d\nu \, dt$$

Integrating over the half-sphere facing the observer:

$$E_\nu = \frac{dA''}{d^2} {R_0}^2 \, d\nu \, dt \int_0^{2\pi} \int_0^{\pi/2} I_\nu(\cos\theta)\cos\theta \, \sin\theta \, d\theta \, d\phi$$
$$= \frac{dA''}{d^2} {R_0}^2 \, d\nu \, dt \, {F_\nu}^+$$

Monochromatic flux received by observer in unit time, per unit area per unit frequency:

$$f_\nu = \frac{{R_0}^2}{d^2} {F_\nu}^+$$



Substituting for that solid angle ($d\Omega$) and expressing the element of area dA in terms of spherical polar coordinates, yields an expression which can be integrated over the hemisphere facing the observer to give the monochromatic flux received by the observer in unit time per unit area per unit frequency. The result deduced in the previous lecture is recovered, namely that the observer's flux is the corresponding flux emergent at the stellar surface scaled by the squared ratio of the stellar radius to the stellar distance.

Specific Intensity Moments

$$J_\nu = \frac{1}{4\pi}\int I_\nu \, d\Omega = \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_{-1}^{+1} I_\nu \, d\mu = \frac{1}{2}\int_{-1}^{+1} I_\nu \, d\mu$$

$$H_\nu = \frac{1}{4\pi}\int I_\nu \, \cos\theta \, d\Omega = \frac{1}{2}\int_{-1}^{+1} I_\nu \, \mu \, d\mu = \frac{{F_\nu}^a}{4}$$

$$K_\nu = \frac{1}{4\pi}\int I_\nu \cos^2\theta \, d\Omega = \frac{1}{2}\int_{-1}^{+1} I_\nu \, \mu^2 \, d\mu$$

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$$K_\nu = \frac{1}{4\pi} \int I_\nu \cos^2\theta \, d\Omega = \frac{1}{2} \int_{-1}^{+1} I_\nu \, \mu^2 \, d\mu$$

Weighted averages of the specific intensity are introduced for further development of radiative transfer theory. General expressions are used to derive special cases, valid in spherical geometry, by expressing the infinitesimal solid angle in terms of the spherical polar coordinates θ and ϕ remembering that $\mu = \cos\theta$. J_ν, H_ν, K_ν are respectively the zeroth, first and second moments of the specific intensity I_ν. J_ν is also referred to as the mean intensity and H_ν as the Eddington Flux.

Energy Density & Radiation Pressure

Energy Density:

$$u_\nu = \frac{\text{radiation energy}}{\text{volume}}$$

$$= \frac{1}{c}\int_0^{4\pi} I_\nu \, d\Omega$$

Radiation Pressure:

$$p_\nu = \frac{\text{force}}{\text{area}}$$

$$= \frac{d\,\text{momentum}\ (= \text{E/c})}{dt}\frac{1}{\text{area}}$$

$$= \frac{1}{c}\int_0^{4\pi} I_\nu \cos^2\theta \, d\Omega$$

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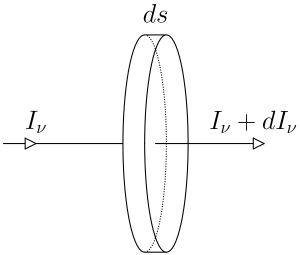
The monochromatic energy density is the radiation energy at frequency ν per unit frequency interval, per unit time, per unit volume. For radiation flow through unit area, the volume traversed in unit time is determined by the velocity of light. The required energy density per unit frequency is therefore the specific intensity integrated over all solid angles and divided by the speed of light. Since the radiation flow is normal to the unit area, $\cos\theta = 1$.

The monochromatic radiation pressure due to radiation emerging at an angle θ with respect to the normal to an infinitesimal area is the rate of change of radiation field momentum in the direction normal to the element of area, divided by its area, which is $(dE_\nu/c)\cos\theta/dA$.

Absorption of Radiation



$$dI_\nu = -\kappa_\nu \, I_\nu \, ds$$

κ_ν : absorption coefficient $\left[\mathrm{cm}^{-1}\right]$

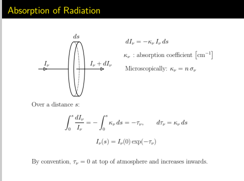
Microscopically: $\kappa_\nu = n \, \sigma_\nu$

Over a distance s:

$$\int_0^s \frac{dI_\nu}{I_\nu} = -\int_0^s \kappa_\nu \, ds = -\tau_\nu, \qquad d\tau_\nu = \kappa_\nu \, ds$$

$$I_\nu(s) = I_\nu(0) \exp(-\tau_\nu)$$

By convention, $\tau_\nu = 0$ at top of atmosphere and increases inwards.



An absorption coefficient κ_ν is expressed in units of inverse length so as to express the decrease in specific intensity (I_ν) on passing through a slab of material of thickness *ds*. Obviously the absorption is directly proportional to the incident radiation in contrast with emission which is independent of it. Microscopically κ_ν will be given by the absorption cross-section (in appropriate units of area) of each absorber, multiplied by the number of absorbers per unit volume (also in appropriate units).

Optical Depth

Optical thickness of a layer determines the specific intensity fraction passing through it.

- If $\tau_\nu = 1, \quad I_\nu(s) = I_\nu(0)/e \simeq 0.37 I_\nu(0)$.
- We can see through an atmosphere to the point where $\tau_\nu \sim 1$.
- Optically thick(thin) medium: $\tau_\nu > (<) 1$.
- $\tau_\nu = 1$ has a geometrical interpretation in terms of the mean free path of photons $\bar{s}$.

$$\tau_\nu = 1 = \int_0^{\bar{s}} \kappa_\nu \, ds$$

- Photons travel on average for a distance $\bar{s}$ before absorption.

Radiative Acceleration

Infinitesimal energy absorbed :

$$
\begin{aligned}
dE_\nu{}^a &= dI_\nu{}^a \cos\theta \, dA \, d\Omega \, dt \, d\nu \\
&= \kappa_\nu \, I_\nu \cos\theta \, dA \, d\Omega \, dt \, d\nu \, ds
\end{aligned}
$$

If κ_ν is isotropic, total energy absorbed is:

$$
\begin{aligned}
E^a &= \int_0^\infty \kappa_\nu \oint I_\nu \cos\theta \, d\Omega \, d\nu \, dA \, ds \, dt \\
&= \int_0^\infty \kappa_\nu \, F_\nu \, d\nu \, dA \, ds \, dt
\end{aligned}
$$

Consequent rate of change of photon momentum $(E/c)/dt$ leads to the radiative acceleration ($g_{\rm rad}$):

$$
\frac{1}{c} \frac{\int_0^\infty \kappa_\nu \, F_\nu \, d\nu}{dt} \, dA \, dt \, ds = g_{\rm rad} \, dm
$$

$$
g_{\rm rad} = \frac{1}{c\rho} \int_0^\infty \kappa_\nu \, F_\nu \, d\nu \qquad dm = \rho \, dA \, ds
$$

Radiative Acceleration

Infinitesimal energy absorbed :

$$dE_\nu^a = dI_\nu^a \cos\theta \, dA \, d\Omega \, dt \, d\nu$$
$$= \kappa_\nu I_\nu \cos\theta \, dA \, d\Omega \, dt \, d\nu \, ds$$

If κ_ν is isotropic, total energy absorbed is:

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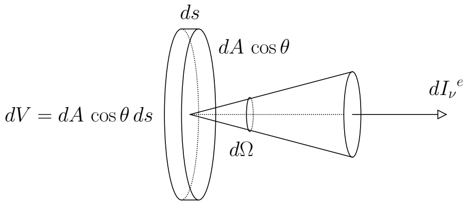
$$= \int_0^\infty \kappa_\nu F_\nu \, d\nu \, dA \, ds \, dt$$

Consequent rate of change of photon momentum $(E/c)/dt$ leads to the radiative acceleration ($g_{\rm rad}$):

$$\frac{1}{c} \frac{\int_0^\infty \kappa_\nu F_\nu \, d\nu}{dt} dA \, dt \, ds = g_{\rm rad} \, dm$$
$$g_{\rm rad} = \frac{1}{c\rho} \int_0^\infty \kappa_\nu F_\nu \, d\nu \qquad dm = \rho \, dA \, ds$$

If the absorption coefficient is isotropic the total energy absorbed by a slab of thickness *ds*, of area *dA* in time *dt* is obtained by integrating over frequency and solid angle. The corresponding rate of change of photon momentum gives the radiative force causing radiative acceleration.

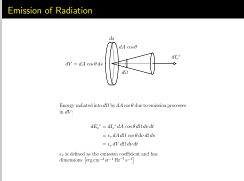
Emission of Radiation



Energy radiated into $d\Omega$ by $dA\cos\theta$ due to emission processes in dV:

$$
\begin{aligned}
dE_\nu{}^e &= dI_\nu{}^e\, dA\ \cos\theta\, d\Omega\, d\nu\, dt \\
&= \epsilon_\nu\, dA\, d\Omega\ \cos\theta\, d\nu\, dt\, ds \\
&= \epsilon_\nu\, dV\ d\Omega\, d\nu\, dt
\end{aligned}
$$

ϵ_ν is defined as the emission coefficient and has dimensions $\left[\mathrm{erg\,cm^{-3}\,sr^{-1}\,Hz^{-1}\,s^{-1}}\right]$



The infinitesimal increment to the monochromatic specific intensity (dE_ν) contributed by emission in a slab of thickness ds is $\epsilon_\nu\, ds$. Note that unlike absorption processes this is independent of the incident specific intensity (I_ν).

Equation of Radiative Transfer

Combine emission and absorption

$$dE_\nu{}^a = dI_\nu{}^a\, dA\, \cos\theta\, d\Omega\, dt\, d\nu = -\kappa_\nu\, I_\nu\, \cos\theta\, dA\, d\Omega\, dt\, d\nu\, ds$$

$$dE_\nu{}^e = dI_\nu{}^e\, dA\, \cos\theta\, d\Omega\, dt\, d\nu = \epsilon_\nu\, \cos\theta\, dA\, d\Omega\, dt\, d\nu\, ds$$

$$\begin{aligned} dE_\nu{}^a + dE_\nu{}^e &= (dI_\nu{}^a + dI_\nu{}^e)\, dA\, \cos\theta\, d\Omega\, d\nu\, dt \\ &= (-\kappa_\nu\, I_\nu + \epsilon_\nu)\, dA\, \cos\theta\, d\Omega\, d\nu\, dt\, ds \end{aligned}$$

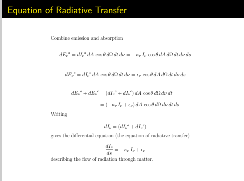
Writing

$$dI_\nu = (dI_\nu{}^a + dI_\nu{}^e)$$

gives the differential equation (the equation of radiative transfer)

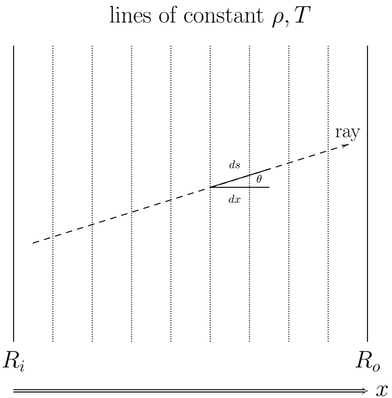
$$\frac{dI_\nu}{ds} = -\kappa_\nu\, I_\nu + \epsilon_\nu$$

describing the flow of radiation through matter.



The equation of radiative transfer is a differential equation giving the rate of change of monochromatic specific intensity (I_ν) with distance (s), due to emission and absorption processes, as radiation is propagated through a slab of gas. Solving the equation of radiative transfer gives $I_\nu(s)$. As presented here, the equation of radiative transfer is one-dimensional and time-independent.

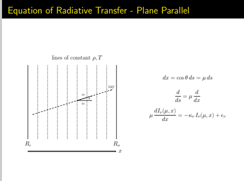
Equation of Radiative Transfer - Plane Parallel



$$dx = \cos\theta\, ds = \mu\, ds$$

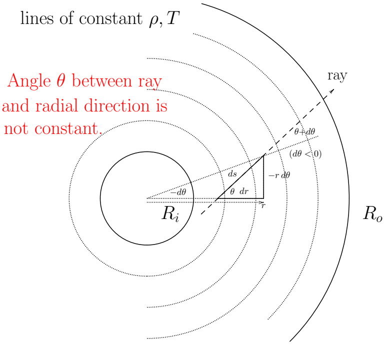
$$\frac{d}{ds} = \mu \frac{d}{dx}$$

$$\mu \frac{dI_\nu(\mu, x)}{dx} = -\kappa_\nu\, I_\nu(\mu, x) + \epsilon_\nu$$



In plane parallel geometry, density and temperature (for example) are constant at any given depth. The thickness of the stellar atmosphere in this approximation is taken to be negligible in comparison with the radius of the star. All quantities are specified in terms of the depth coordinate (x) and a simple transformation allows a monochromatic specific intensity increment to be expressed in terms of x and $\mu = \cos\theta$.

Equation of Radiative Transfer - Spherical



$$\frac{d}{ds} = \frac{dr}{ds}\frac{\partial}{\partial r} + \frac{d\theta}{ds}\frac{\partial}{\partial\theta}$$

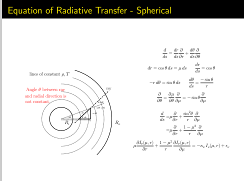
$$dr = \cos\theta\, ds = \mu\, ds \qquad \frac{dr}{ds} = \cos\theta$$

$$-r\, d\theta = \sin\theta\, ds \qquad \frac{d\theta}{ds} = \frac{-\sin\theta}{r}$$

$$\frac{\partial}{\partial\theta} = \frac{\partial\mu}{\partial\theta}\frac{\partial}{\partial\mu} = -\sin\theta\frac{\partial}{\partial\mu}$$

$$\begin{aligned}\frac{d}{ds} &= \mu\frac{\partial}{\partial r} + \frac{\sin^2\theta}{r}\frac{\partial}{\partial\mu}\\ &= \mu\frac{\partial}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial}{\partial\mu}\end{aligned}$$

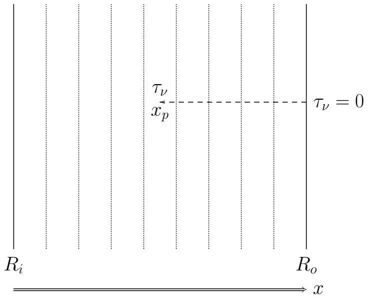
$$\mu\frac{\partial I_\nu(\mu,r)}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial I_\nu(\mu,r)}{\partial\mu} = -\kappa_\nu\, I_\nu(\mu,r) + \epsilon_\nu$$



In spherical geometry the angle θ between a ray and the radial direction is not constant. As a consequence, the equation of radiative transfer is more complicated than in the plane-parallel case.

Optical Depth & Source Function

Optical depth increasing towards interior.



$$-\kappa_\nu \, dx = d\tau_\nu \qquad \tau_\nu = -\int_{R_o}^{x_p} \kappa_\nu \, dx$$

For a plane-parallel atmosphere

$$\mu \frac{dI_\nu(\mu, x)}{dx} = -\kappa_\nu(x)\, I_\nu(\mu, x) + \epsilon_\nu$$

gives

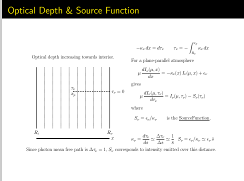
$$\mu \frac{dI_\nu(\mu, \tau_\nu)}{d\tau_\nu} = I_\nu(\mu, \tau_\nu) - S_\nu(\tau_\nu)$$

where

$S_\nu = \epsilon_\nu / \kappa_\nu$ is the SourceFunction.

$$\kappa_\nu = \frac{d\tau_\nu}{ds} \simeq \frac{\Delta\tau_\nu}{\Delta s} \simeq \frac{1}{\bar{s}} \quad S_\nu = \epsilon_\nu / \kappa_\nu \simeq \epsilon_\nu \, \bar{s}$$

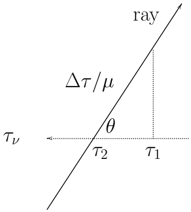
Since photon mean free path is $\Delta\tau_\nu = 1$, S_ν corresponds to intensity emitted over this distance.



Monochromatic optical depth (τ_ν) is introduced as minus the integral of the appropriate absorption coefficient over the corresponding interval in atmospheric height, where the minus sign is needed because optical depth increases as height in the atmosphere decreases. Then it is natural to express the ratio ϵ_ν/κ_ν as the source function (S_ν) and reformulate the equation of radiative transfer accordingly.

Radiative Transfer Equation – Formal Solution

For a plane-parallel atmosphere:



$$\mu \frac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu$$

$$e^{-\tau_\nu/\mu}\,\mu\,\frac{dI_\nu}{d\tau_\nu} = I_\nu\,e^{-\tau_\nu/\mu} - S_\nu\,e^{-\tau_\nu/\mu}$$

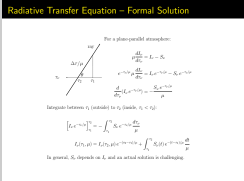
$$\frac{d}{d\tau_\nu}(I_\nu\,e^{-\tau_\nu/\mu}) = -\frac{S_\nu\,e^{-\tau_\nu/\mu}}{\mu}$$

Integrate between τ_1 (outside) to τ_2 (inside, $\tau_1 < \tau_2$):

$$\left[I_\nu\,e^{-\tau_\nu/\mu}\right]_{\tau_1}^{\tau_2} = -\int_{\tau_1}^{\tau_2} S_\nu\,e^{-\tau_\nu/\mu}\,\frac{d\tau_\nu}{\mu}$$

$$I_\nu(\tau_1,\mu) = I_\nu(\tau_2,\mu)\,e^{-(\tau_2-\tau_1)/\mu} + \int_{\tau_1}^{\tau_2} S_\nu(t)\,e^{-(t-\tau_1)/\mu}\,\frac{dt}{\mu}$$

In general, S_ν depends on I_ν and an actual solution is challenging.



A formal solution of the radiative transfer equation for plane parallel geometry, using optical depth as the independent variable, may be obtained by multiplying by $e^{-\tau_\nu/\mu}$ and collecting terms in I_ν on the left-hand side and S_ν on the right-hand side. The left-hand side is easily integrated between two optical depths to give the monochromatic specific intensity at one optical depth in terms of the monochromatic specific intensity at the other optical depth and some integral involving the monochromatic source function. In general S_ν is dependent on I_ν and further progress is a challenge

Radiative Transfer Equation – Boundary Conditions

For incoming radiation $\mu < 0$ and inward radiation from outside is usually neglected: $I_\nu(\tau_\nu = 0, \mu < 0) = 0$

$$I_\nu^{\rm in}(\tau_\nu, \mu) = \int_{\tau_\nu}^{0} S_\nu(t)\, e^{-(t-\tau_\nu)/\mu}\, \frac{dt}{\mu}$$

For outgoing radiation $\mu > 0$ and we have either

- a finite slab or shell for which $I_\nu(\tau_{\rm max}, \mu) = I_\nu^+(\mu)$ or
- a semi-infinite (planar or spherical) case where $\lim_{\tau_\nu \to \infty} I_\nu(\tau_\nu, \mu)\, e^{-\tau_\nu/\mu} = 0$

The second applies in the case of stellar atmospheres and so

$$I_\nu^{\rm out}(\tau_\nu, \mu) = \int_{\tau_\nu}^{\infty} S_\nu(t)\, e^{-(t-\tau_\nu)/\mu}\, \frac{dt}{\mu}$$

For the case where $\tau_\nu = 0$, the above equation gives the emergent intensity:

$$I_\nu(0, \mu) = \int_{0}^{\infty} S_\nu(t)\, e^{-t/\mu}\, \frac{dt}{\mu}$$

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From any point in a stellar atmosphere, the gas becomes opaque ($\tau_\nu \to \infty$ at all frequencies in the direction of the stellar centre. As a result, radiation is only received from the surface layers; this provides the boundary condition in so far as the specific intensity at any finite optical depth is determined by an integral involving the source function between that optical depth and infinity.

Source Function – Simple Cases – I

- In Local Thermodynamic Equilibrium, photons are absorbed and re-emitted at the local temperature (T) (Kirchhoff's Law)

$$S_\nu = \frac{\epsilon_\nu}{\kappa_\nu} = B_\nu(T).$$

- For coherent isotropic scattering, absorption is characterised by the scattering coefficient σ_ν, analogous to κ_ν:

$$dI_\nu = -\sigma_\nu \, I_\nu \, ds \qquad dE_\nu{}^{\rm em} = \oint_{4\pi} \epsilon_\nu{}^{\rm sc} \, d\Omega \qquad dE_\nu{}^{\rm abs} = \oint_{4\pi} \sigma_\nu{}^{\rm sc} \, I_\nu \, d\Omega.$$

At each ν, $dE_\nu{}^{\rm em} = dE_\nu{}^{\rm abs}$:

$$\oint_{4\pi} \epsilon_\nu{}^{\rm sc} \, d\Omega = \oint_{4\pi} \sigma_\nu I_\nu \, d\Omega$$

$$\epsilon_\nu{}^{\rm sc} \oint_{4\pi} d\Omega = \sigma_\nu \oint_{4\pi} I_\nu \, d\Omega$$

$$\frac{\epsilon_\nu{}^{\rm sc}}{\kappa_\nu} = \frac{1}{4\pi} \oint_{4\pi} I_\nu \, d\Omega$$

$$S_\nu = J_\nu.$$

Source function is completely dependent on radiation field and independent of T.

Source Function – Simple Cases – II

- Mixed case:

$$S_\nu = \frac{\epsilon_\nu + {\epsilon_\nu}^{\rm sc}}{\kappa_\nu + \sigma_\nu}$$

$$= \frac{\kappa_\nu}{\kappa_\nu + \sigma_\nu}\frac{\epsilon_\nu}{\kappa_\nu} + \frac{\sigma_\nu}{\kappa_\nu + \sigma_\nu}\frac{{\epsilon_\nu}^{\rm sc}}{\sigma_\nu}$$

$$= \frac{\kappa_\nu}{\kappa_\nu + \sigma_\nu}B_\nu + \frac{\sigma_\nu}{\kappa_\nu + \sigma_\nu}J_\nu$$

Source Function – Simple Cases – II

- Mixed case:

$$S_\nu = \frac{\epsilon_\nu + \epsilon_\nu{}^{\rm sc}}{\kappa_\nu + \sigma_\nu} = \frac{\kappa_\nu}{\kappa_\nu + \sigma_\nu}\frac{\epsilon_\nu}{\kappa_\nu} + \frac{\sigma_\nu}{\kappa_\nu + \sigma_\nu}\frac{\epsilon_\nu{}^{\rm sc}}{\sigma_\nu} = \frac{\kappa_\nu}{\kappa_\nu + \sigma_\nu} B_\nu + \frac{\sigma_\nu}{\kappa_\nu + \sigma_\nu} J_\nu$$

Simple solutions to the radiative transfer equation arise when photons are absorbed and re-emitted at the local temperature so that the source function (S_ν) is equal to the Planck Function (B_ν). A second simple case is where absorption is absent and photons are only scattered isotropically and coherently; in this case the source function is the mean intensity (J_ν). In the mixed case, the contributions to S_ν from absorption and isotropic coherent scattering are separated. Because J_ν depends on I_ν, some iterative scheme is needed to solve the transfer equation in all but the simplest case when $S_\nu = B_\nu$.

Diffusion Approximation – I

At large optical depths ($\tau_\nu \gg 1$) and photons are local so that $S_\nu \to B_\nu$. Expanding as a power series about τ_ν:

$$S_\nu(t) = B_\nu(t) = \sum_{n=0}^{\infty} \frac{d^n B_\nu(\tau_\nu)}{d{\tau_\nu}^n}(t - \tau_\nu)^n/n!$$

Photons are local and therefore $(t - \tau_\nu) \sim 0$, justifying the retention of only the first order term (Diffusion Approximation):

$$B_\nu(t) = B_\nu(\tau_\nu) + \frac{dB_\nu}{d\tau_\nu}(t - \tau_\nu)$$

$$I_\nu^{\text{out}}(\tau_\nu, \mu) = \int_{\tau_\nu}^{\infty} S_\nu(t)\, e^{-(t-\tau_\nu)/\mu}\, \frac{dt}{\mu}$$

$$= \int_{\tau_\nu}^{\infty} \left[B_\nu(\tau_\nu) + \frac{dB_\nu}{d\tau_\nu}(t - \tau_\nu)\right] e^{-(t-\tau_\nu)/\mu}\, \frac{dt}{\mu}$$

Diffusion Approximation – II

Let

$$u = \frac{t - \tau_\nu}{\mu} \qquad \rightarrow \qquad dt = \mu \, du$$

and since

$$\int_0^\infty u^k \, e^{-u} \, du = k!$$

$$I_\nu^{\rm out}(\tau_\nu, \mu) = \int_{\tau_\nu/\mu}^\infty \left[B_\nu(t) + \frac{dB_\nu}{d\tau_\nu} \mu \, u \right] e^{-u} \, du$$

$$= B_\nu(t) + \mu \, \frac{dB_\nu}{d\tau_\nu}$$

$$I_\nu^{\rm in}(\tau_\nu, \mu) = - \int_0^{\tau_\nu/\mu} \left[B_\nu(t) + \frac{dB_\nu}{d\tau_\nu} \mu \, u \right] e^{-u} \, du$$

Eddington-Barbier relation for observed emergent intensity obtained for $\tau_\nu = 0$; it depends linearly on μ.

Diffusion Approximation – II

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$$I_\nu^{\rm in}(\tau_\nu, \mu) = -\int_0^{\tau_\nu/\mu} \left[B_\nu(t) + \frac{dB_\nu}{d\tau_\nu} \mu\, u \right] e^{-u}\, du$$

Eddington-Barbier relation for observed emergent intensity obtained for $\tau_\nu = 0$; it depends linearly on μ.

The diffusion approximation provides a robust solution to the radiative transfer equation at large optical depths, as appropriate for stellar interiors, when photons are necessarily local. It is supposed that temperature is monotonically increasing with optical depth (τ_ν) and so the Planck Function (B_ν) may, in this context, be regarded as a function of τ_ν. The approach is then to expand B_ν in a Taylor's Series about the local τ_ν and as the optical depth range is small, only the first order term need be retained. The standard integral is Equation 3.384 in Gradshteyn & Ryzhik.

Eddington Approximation

In a planar atmosphere with

$$I_\nu = B_\nu + \mu \frac{dB_\nu}{d\tau_\nu}$$

we have:

$$J_\nu = \frac{1}{2} \int_{-1}^{+1} I_\nu \, d\mu = \frac{1}{2} \left[\mu \, B_\nu + \frac{\mu^2}{2} \frac{dB_\nu}{d\tau_\nu} \right]_{-1}^{+1} = B_\nu(\tau_\nu)$$

$$H_\nu = \frac{{F_\nu}^a}{4} = \frac{1}{2} \int_{-1}^{+1} I_\nu \, \mu \, d\mu = \frac{1}{2} \left[\frac{\mu^2}{2} B_\nu + \frac{\mu^3}{3} \frac{dB_\nu}{d\tau_\nu} \right]_{-1}^{+1}$$

$$= \frac{1}{3} \frac{dB_\nu}{d\tau_\nu} = -\frac{1}{3} \frac{1}{\kappa_\nu} \frac{dB_\nu}{dx} = -\frac{1}{3} \frac{1}{\kappa_\nu} \frac{dB_\nu}{dT} \frac{dT}{dx}$$

$$K_\nu = \frac{1}{2} \int_{-1}^{+1} I_\nu \, \mu^2 \, d\mu = \frac{1}{2} \left[\frac{\mu^3}{3} B_\nu + \frac{\mu^4}{4} \frac{dB_\nu}{d\tau_\nu} \right]_{-1}^{+1} = \frac{1}{3} B_\nu(\tau_\nu)$$

Eddington Approximation

In a planar atmosphere with

$$I_\nu = B_\nu + \mu \frac{dB_\nu}{d\tau_\nu}$$

we have:

$$J_\nu = \frac{1}{2}\int_{-1}^{+1} I_\nu \, d\mu = \frac{1}{2}\left[\mu B_\nu + \frac{\mu^2}{2}\frac{dB_\nu}{d\tau_\nu}\right]_{-1}^{+1} = B_\nu(\tau_\nu)$$

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Using the Diffusion Approximation, simple expressions follow for J_ν, H_ν and K_ν.

Schwarzschild-Milne Equations – I

$$J_\nu = \frac{1}{2}\int_{-1}^{+1} I_\nu \, d\mu = \frac{1}{2}\int_{0}^{+1} I_\nu{}^{\text{out}} \, d\mu + \frac{1}{2}\int_{-1}^{0} I_\nu{}^{\text{in}} \, d\mu$$

$$= \frac{1}{2}\left[\int_0^1 \int_{\tau_\nu}^{\infty} S_\nu(t)\, e^{-(t-\tau_\nu)/\mu} \frac{dt}{\mu} d\mu \; - \int_{-1}^{0}\int_0^{\tau_\nu} S_\nu(t)\, e^{-(t-\tau_\nu)/\mu} \frac{dt}{\mu} d\mu\right]$$

$$= \frac{1}{2}\left[\int_1^{\infty} \int_{\tau_\nu}^{\infty} S_\nu(t)\, e^{-(t-\tau_\nu)w}\, dt\, \frac{dw}{w} \; + \int_{1}^{\infty}\int_0^{\tau_\nu} S_\nu(t)\, e^{-(\tau_\nu-t)w}\, dt\, \frac{dw}{w}\right]$$

$$= \frac{1}{2}\left[\int_{\tau_\nu}^{\infty} S_\nu(t) \int_1^{\infty} e^{-(t-\tau_\nu)w} \frac{dw}{w} dt \; + \int_0^{\tau_\nu} S_\nu(t) \int_1^{\infty} e^{-(\tau_\nu-t)w} \frac{dw}{w} dt\right]$$

Where $w = 1/\mu$ and $w = -1/\mu$ for left and right double integrals respectively; in both cases $dw/w = -d\mu/\mu$. Since both exponents are greater than zero:

$$J_\nu = \frac{1}{2}\int_0^{\infty} S_\nu(t) \int_1^{\infty} e^{-w|t-\tau_\nu|} \frac{dw}{w} dt$$

$$= \frac{1}{2}\int_0^{\infty} S_\nu(t)\, E_1(|t-\tau_\nu|)\, dt$$

Schwarzschild-Milne Equations – II

Introducing the Λ-Operator:

$$\Lambda_{\tau_\nu} = \frac{1}{2}\int_0^\infty f(t)\, E_1(|t-\tau_\nu|)\, dt$$

$$J_\nu(\tau_\nu) = \Lambda_{\tau_\nu}\left[S_\nu(t)\right]$$

Similarly for the other two specific intensity moments:

$$H_\nu(\tau_\nu) = \frac{1}{2}\int_{\tau_\nu}^\infty S_\nu(t)\, E_2(|t-\tau_\nu|)\, dt \; - \frac{1}{2}\int_0^{\tau_\nu} S_\nu(t)\, E_2(|\tau_\nu - t|)\, dt$$

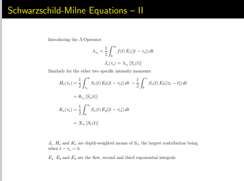
$$= \Phi_{\tau_\nu}\left[S_\nu(t)\right]$$

$$K_\nu(\tau_\nu) = \frac{1}{2}\int_0^\infty S_\nu(t)\, E_3(|t-\tau_\nu|)\, dt$$

$$= X_{\tau_\nu}\left[S_\nu(t)\right]$$

J_ν, H_ν and K_ν are depth-weighted means of S_ν, the largest contribution being when $t - \tau_\nu = 0$.

E_1, E_2 and E_3 are the first, second and third exponential integrals.



In plane parallel geometry, the zeroth, first and second moments of the radiation field may be expressed as integrals over the source function multiplied respectively by the first, second and third exponential integrals. The first exponential integral can be obtained from a published Chebyshev series; from this the second and higher order exponential integrals may be obtained using the usual recurrence formula.

Lecture 2: Summary

Essential points covered in second lecture:

- Specific intensity defined and its invariance, in the absence of absorption, verified.
- It was shown how specific intensity is related to radiative flux, luminosity and observed flux.
- Energy density, radiation pressure and the absorption of radiation were discussed.
- The equation of radiative transfer, optical depth and source function were introduced. Simple special case solutions of the transfer equation were presented with formal solution and boundary conditions.
- Diffusion approximation needed for stellar structure and evolution calculations was derived.
- Schwarzschild-Milne equations were also derived as these are needed for stellar atmosphere and synthetic spectrum calculations.

Stellar evolution depends on initial photospheric abundances and their determination from spectra are to be discussed in the next lecture.

Acknowledgement

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