Introduction – I

Main physical processes which determine the structure of stars:

- Stars are held together by gravitation — attraction exerted on each part of the star by all other parts.
- Collapse is resisted by internal thermal pressure.
- Gravitation and internal thermal pressure must be (at least almost) in balance.
- Stars continually radiate into space; for thermal properties to be constant, a continuous energy source must exist.
- Theory must describe origin of energy and its transport to the surface.

Two fundamental assumptions are made:

- Neglect the rate of change of properties due to stellar evolution in the first instance; assume these are constant with time.
- All stars are spherical and symmetric about their centres of mass.

Introduction – II

For stars which are isolated, static and spherically symmetric, there are four equations to describe structure. All physical quantities depend only on distance from the centre of the star:

- Equation of Hydrostatic Equilibrium – at each radius, forces due pressure difference balance gravity,
- Conservation of mass,
- Conservation of energy – at each radius, the change in the energy flux is the local rate of energy release and
- Equation of Energy Transport – the relation between the energy flux and the local temperature gradient.

These basic equations are supplemented with:

- an equation of state giving gas pressure as a function of its density and temperature,
- opacity (how opaque the gas is to the radiation field) and
- core nuclear energy generation rate.

Hydrostatic Equilibrium – I

In connection with a discussion of radial pulsation and subsequently while considering stellar atmospheres, the equation

$$\frac{dP}{dr} = -\frac{Gm(r)}{r^2}\rho(r)$$

was derived; it is the Equation of Hydrostatic Equilibrium (or Hydrostatic Support) and is the **First Equation of Stellar Structure**.

When the pressure gradient is not exactly balanced by gravity, the equation of motion was found to be

$$\frac{dP}{dr} = -\frac{Gm(r)}{r^2}\rho(r) - \rho(r)\frac{d^2r}{dt^2}.$$

Near the surface $Gm(r)/r^2 = g$ and suppose we then have a small inward acceleration of $d^2r/dt^2 = \beta g$, the displacement after time t if it begins at rest is

$$d = \frac{1}{2}\beta g t^2.$$

Hydrostatic Equilibrium – II

If we allowed the star to collapse, that is set $d = R$ where R is the stellar radius, then the time required would be

$$t = \left(\frac{2R}{\beta g}\right)^{1/2} = \frac{1}{\sqrt{\beta}}\left(\frac{2R^3}{GM}\right)^{1/2}.$$

In the absence of a pressure gradient $dP/dr = 0$, $\beta \simeq 1$ and we have an expression for the stellar dynamical time

$$t_d = \left(\frac{2R^3}{GM}\right)^{1/2},$$

which provides an estimate for the characteristic period on which a stellar interior vibrates in response to small mechanical disturbances such as flares, convection or impacts; it is roughly the time required for a sound wave to cross the star.

In the case of the Sun $R_\odot \simeq 7 \times 10^8\,\mathrm{m}$ and $M_\odot \simeq 2 \times 10^{30}\,\mathrm{kg}$ giving $t_d \simeq 2300\,\mathrm{seconds}$.

Hydrostatic Equilibrium – III

Stars are rotating gaseous bodies and is consequent flattening at the poles so significant that departures from spherical symmetry need to be taken into account?

Consider δm to be a mass near the surface of a star of mass M and radius R; the inward centripetal force acting on it to provide circular motion is $\delta m\,\omega^2\,R$ where ω is the angular rotation velocity of the star.

Condition for no significant departure from spherical symmetry is:

$$\delta m\,\omega^2\,R \ll \frac{G\,M\,\delta m}{R^2} \quad \text{or} \quad \omega^2 \ll \frac{GM}{R^3} \quad \text{or} \quad \omega^2 \ll \frac{2}{{t_d}^2}.$$

As $\omega = 2\,\pi/t_{\mathrm{rot}}$, where t_{rot} is the stellar rotation period, spherical symmetry exists if $t_{\mathrm{rot}} \gg t_d$.

For the Sun $t_{\mathrm{rot}} \sim 1\,\text{month}$ and $t_d \sim 0.5\,\text{hours}$ and so departures from spherical symmetry can be ignored, as is the case for most stars.

Some stars do rotate rapidly and rotational effects must be included in the structure equations.

Mass Conservation

In connection with a discussion of radial pulsation, the equation

$$\frac{dM(r)}{dr} = 4\,\pi\,r^2\,\rho(r)$$

was derived; it is the Equation of Mass Conservation and is the **Second Equation of Stellar Structure**.

Minimum Central Pressure – I

Hydrostatic equilibrium and mass conservation

$$\frac{dP(r)}{dr} = -\frac{G\,M(r)\,\rho(r)}{r^2} \quad \text{and} \quad \frac{dM(r)}{dr} = 4\,\pi\,r^2\,\rho(r)$$

are two of the four equations of stellar structure discussed.

Dividing these two equations gives

$$\frac{dP(r)}{dM(r)} = -\frac{G\,M(r)}{4\,\pi r^4}$$

which on integrating over the whole star results in

$$P_{\mathrm{c}} - P_{\mathrm{s}} = \int_0^M \frac{GM(r)}{4\,\pi r^4}\,dM(r)$$

where P_{c} and P_{s} are pressures at the stellar centre and surface respectively.

Minimum Central Pressure – II

Clearly the maximum value of r is at the stellar surface when $r = R$; here the integrand is a minimum and therefore

$$P_c - P_s = \int_0^M \frac{GM(r)}{4\,\pi r^4}\,dM(r) > \int_0^M \frac{GM(r)}{4\,\pi R^4}\,dM(r) = \frac{GM^2}{8\,\pi R^4}.$$

To a good approximation, pressure at the stellar surface is zero giving a minimum value for the central pressure of

$$P_c > \frac{GM^2}{8\,\pi R^4}\,.$$

For example, for the Sun

$$P_c > 4.5 \times 10^{13} \quad \mathrm{Nm}^{-2}$$

which indicates that gas at the centre of the Sun is not an ordinary gas typically found in the Earth's atmosphere.

Virial Theorem – I

Hydrostatic equilibrium and mass conservation

$$\frac{dP(r)}{dr} = -\frac{G\,M(r)\,\rho(r)}{r^2} \quad \text{and} \quad \frac{dM(r)}{dr} = 4\,\pi\,r^2\,\rho(r)$$

are two of the four equations of stellar structure discussed.

Dividing these two equations gives

$$\frac{dP(r)}{dM(r)} = -\frac{G\,M(r)}{4\,\pi r^4}$$

which on multiplying both sides by $4\pi r^3$ and integrating over the whole star results in

$$4\pi r^3\,dP(r) = \frac{GM(r)}{r}\,dM(r)$$

$$3\int_{P_c}^{P_s} V(r)\,dP(r) = -\int_0^M \frac{GM(r)}{r}\,dM(r)$$

where $V(r)$ is the volume contained within radius r.

Virial Theorem – II

Integrating the left hand side by parts gives

$$3\,[PV]_{P_c}^{P_s} - 3\int_{V_c}^{V_s} P(r)\,dV(r) = -\int_0^M \frac{GM(r)}{r}\,dM(r).$$

Clearly $V_c = 0$ and $P_s = 0$ and therefore

$$3\int_0^{V_s} P(r)\,dV(r) - \int_0^M \frac{GM(r)}{r}\,dM(r) = 0 \quad \text{and}$$

$$3\int_0^{V_s} P(r)\,dV(r) + \boldsymbol{\Omega} = 0.$$

The second equation is the Virial Theorem, where

$$-\boldsymbol{\Omega} = \int_0^M \frac{GM(r)}{r}\,dM(r),$$

and is of great importance in astrophysics. $\boldsymbol{\Omega}$ is the total gravitational energy of the star or the energy released in forming the star from its components dispersed to infinity. As a consequence, the left-hand term is the thermal energy of the star.

Mean Temperature – I

For all points inside the star, $r < R$ and so

$$-\mathbf{\Omega} = \int_0^M \frac{GM(r)}{r}\, dM(r) > \int_0^M \frac{GM(r)}{R}\, dM(r) = \frac{GM^2}{2R}\,.$$

Now $dM = \rho\, dV$ and the Virial Theorem can be written as

$$-\mathbf{\Omega} = 3\int_0^{V_s} P(r)\, dV(r) = 3\int_0^M \frac{P(r)}{\rho(r)}\, dM(r).$$

Assume for the moment that stars are composed of an ideal gas and that radiation pressure is negligible; in this case:

$$P = P_{\mathrm{gas}} = \frac{k}{m_{\mathrm{H}}\bar{\mu}}\,\rho\, T$$

and so

$$-\mathbf{\Omega} = 3\int_0^M \frac{k\, T(r)}{m_{\mathrm{H}}\bar{\mu}}\, dM(r) > \frac{GM^2}{2R}$$

and

$$\int_0^M T(r)\, dM(r) > \frac{G\, M^2\, m_{\mathrm{H}}\, \bar{\mu}}{6\, k\, R}.$$

Mean Temperature – II

The left-hand-side of the above equation can be thought of as the sum of the mass-elements $dM(r)$ which make up the star; dividing this by the total mass of the star gives its mean temperature:

$$M\,\bar{T} = \int_0^M T(r)\,dM(r)$$

$$\bar{T} > \frac{G\,M\,m_{\rm H}\,\bar{\mu}}{6\,k\,R}.$$

As an example, for the Sun we have

$$\bar{T} > 4 \times 10^6\,\bar{\mu}\,{\rm K}$$

Hydrogen is generally (but not always) the most abundant element in stars and for fully ionised hydrogen $\bar{\mu} = 1/2$

$$\bar{T}_\odot > 2 \times 10^6\,{\rm K}.$$

Physical State of Stellar Material – I

The mean density of material in the Sun is

$$\bar{\rho}_\odot = \frac{3M_\odot}{4\pi {R_\odot}^3} = 1.4 \times 10^3 \, \mathrm{kg \ m^{-3}}$$

which is a little higher than water and other ordinary liquids. Such liquids become gaseous at $T \ll \bar{T}_\odot$.

Moreover, the kinetic energy of particles at $\bar{T}_\odot$ is much higher than the ionisation potential of hydrogen. Thus the gas must be almost completely ionised and exist as a plasma; it can therefore withstand greater compression without departure from an ideal gas state.

Note that an ideal gas demands that the distances between particles are much larger that their sizes. A fully ionised plasma satisfies the condition more readily than a neutral gas because nuclear dimensions are $\sim 10^{-15}$ m whereas atomic dimensions are $\sim 10^{-10}$ m.

Physical State of Stellar Material – II

In order to assess the importance of radiation pressure, recall that radiation in the frequency interval $\nu \to \nu + d\nu$ contributes

$$p_\nu = \frac{1}{c} \oint_{4\pi} I_\nu \cos^2\theta \, d\Omega = 4\pi \, \frac{K_\nu}{c}$$

to the radiation pressure.

In the Eddington Approximation, $K_\nu = B_\nu(T)/3$ and the total radiation pressure is

$$p = \frac{4\pi}{3c} \int_0^\infty B_\nu(T) \, d\nu$$

$$= \frac{4\pi}{3c} \int_0^\infty \frac{2h\nu^3}{c^2} \, \frac{1}{\exp(h\nu/kT) - 1} \, d\nu$$

Physical State of Stellar Material – III

Let $x = h\nu/kT$ and then $d\nu = (kT/h)\,dx$ so that

$$p = \frac{4\pi}{3c}\frac{2\,k^4\,T^4}{c^2\,h^3}\int_0^\infty \frac{x^3}{\exp(x)-1}\,dx$$

$$= \frac{1}{3}\frac{8\,k^4\,T^4}{c^3\,h^3}\frac{\pi^5}{15}$$

$$= \frac{1}{3}a\,T^4$$

where $a = (8k^4\pi^5/15c^3h^3)$ is known as the **radiation constant**.

Now comparing gas and radiation pressures gives:

$$\frac{P_{\rm rad}}{P_{\rm gas}} = \frac{aT^4}{3}\left(\frac{kT\rho}{m_{\rm H}\bar{\mu}}\right)^{-1} = \frac{m_{\rm H}\bar{\mu}aT^3}{3k\rho}$$

Taking $T \sim \bar{T}_\odot = 2\times10^6\,{\rm K}$, $\rho \sim \bar{\rho}_\odot = 1.4\times10^3\,kg\,m^{-3}$, $m_{\rm H} = 1.67\times10^{-27}\,{\rm kg}$, $\bar{\mu} = 0.5$ and $a = 7.6\times10^{-16}{\rm Nm^{-2}K^{-4}}$ gives $P_{\rm rad}/P_{\rm gas} \simeq 10^{-4}$.

Significance of Radiation Pressure

Hence radiation pressure appears to be negligible at an average point in the Sun. Without recourse to our knowledge of how energy is generated in the Sun, a value for its internal temperature has been derived and it has been deduced that the material is essentially an ideal gas with negligible radiation pressure.

Now we consider higher mass stars by replacing density in the pressure ratio expression giving

$$\frac{P_{\rm rad}}{P_{\rm gas}} = \frac{m_{\rm H}\bar{\mu}aT^3}{3k\left(\frac{3M}{4\pi R^3}\right)} = \frac{4\pi\, m_{\rm H}\,\bar{\mu}\,a}{9k}\,\frac{R^3\,T^3}{M}$$

From the Virial Theorem $\bar{T}$ scales as M/R and so

$$\frac{P_{\rm rad}}{P_{\rm gas}} \propto M^2.$$

Clearly radiation pressure becomes increasingly significant as stellar mass increases.

Energy Generation in the Sun

Only considered the dynamical properties of a star and the state of stellar material. Attention now needs to be given to the source of stellar energy.

Energy is converted from some form in which it is not immediately available into a form which can be radiated.

How much energy does the Sun need to generate in order to maintain its observed luminosity?

$$L_\odot = 4 \times 10^{26}\, W$$

and it is known from geological records that the Sun has not changed appreciably in 10^9 years (3×10^{16} seconds) and so it has radiated $1.2 \times 10^{43} J$. To get the equivalent mass lost, divide by c^2 giving

$$M_{\rm lost} = \frac{1.2 \times 10^{43}}{9.0 \times 10^{16}} = 1.3 \times 10^{26}\ {\rm kg} \simeq 10^{-4}\, M_\odot.$$

Source of Energy Generation – I

Consider four possible sources of energy generation in stars:

- cooling,
- contraction,
- chemical reactions and
- nuclear reactions.

Cooling and contraction are closely related and are considered together. Cooling is the simplest idea. Suppose the radiative energy of the Sun is due to the Sun being hotter when it was formed and has since been cooling down. Or since its formation, the Sun has been slowly contracting with the consequent release of gravitational potential energy which is converted to radiation.

Source of Energy Generation – II

In an ideal gas, the thermal energy of a particle having three degrees of freedom is $(3/2)kT$. Total thermal energy per unit volume is then $(3/2)nkT$, where n is the number of particles per unit volume.

From the Virial Theorem

$$3\int_0^{V_s} P\,dV + \mathbf{\Omega} = 0$$

assuming stellar material to be an ideal gas

$$3\int_0^{V_s} nkT\,dV + \mathbf{\Omega} = 0.$$

Define U to be the integral over volume of the thermal energy per unit volume; then

$$2U + \mathbf{\Omega} = 0.$$

The negative gravitational energy of a star is equal to twice its thermal energy. The time for which the present thermal energy of the Sun could supply its luminosity, and the time for which a previous release of gravitational energy could have done so, differ by a factor of two.

Source of Energy Generation – III

As already shown, if stellar material is a perfect gas

$$-\mathbf{\Omega} > \frac{GM^2}{2R}$$

and so adopting

$$-\mathbf{\Omega} \sim \frac{GM^2}{2R}$$

as a crude approximation, it can be seen that the conversion of gravitational potential energy to radiation would provide a luminosity L for a time

$$t_{\rm th} = \frac{GM^2}{LR},$$

where $t_{\rm th}$ is defined as the thermal (or Kelvin-Helmholtz) time-scale.

Substituting values for the Sun gives $t_{\rm th\odot} = 3 \times 10^7$ years. Therefore if the Sun were powered by either contraction or cooling, it would have changed substantially in the last 10^7 years, a factor of ~ 100 too short to account for age constraints imposed by fossil and geological records.

Source of Energy Generation – IV

Chemical reactions can be ruled out as a possible source for stellar energy because:

- It was shown above that the Sun is made up of largely ionised material. Hydrogen in particular is almost completely ionised except in the atmosphere. There are therefore very few atoms or ions having the bound electrons needed for chemical reactions to proceed.
- The energy source needs to provide, in the solar case, the energy equivalent of at least $10^{-4}\,M_\odot$ over $\sim 10^9$ years. Chemical reactions such as the combustion of fossil fuels release the energy equivalent of $\sim 5 \times 10^{-10}\,M_\odot$ in the same period.

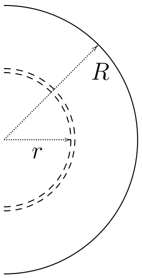
Source of Energy Generation – V

Therefore nuclear reactions need to be considered as the only known viable way of producing sufficiently large amounts of energy to power a star. Consider the fusion of four protons (atomic mass 1.008172) into ^{4}He (an alpha particle of atomic mass 4.003875):

- Mass deficit with each fusion reaction is 0.0288 atomic mass units.
- The equivalent in kilograms is $0.0288 \times (1.67 \times 10^{-27}) = 4.8 \times 10^{-29}$ kg.
- Number of reactions required before $10^{-4}\,M_\odot$ are converted into energy is $((2.0 \times 10^{30}) \times 10^{-4})/(4.8 \times 10^{-29}) = 4.16 \times 10^{54}$.
- Energy equivalent of each fusion reaction mass-deficit is $(4.8 \times 10^{-29}) \times (3 \times 10^{8})^2 = 4.33 \times 10^{-12}$ J.
- Therefore the total energy produced over 10^9 years by all fusion reactions is $(4.33 \times 10^{-12}) \times (4.16 \times 10^{54}) = 1.8 \times 10^{43}$ J.
- As 10^9 years $\simeq 3 \times 10^{16}$ seconds the consequent stellar luminosity would be 6×10^{26} W which is within a factor of 2 of the observed solar luminosity.

Provided conditions allow the fusion of hydrogen into helium to proceed, it is a probable energy source in the case of the Sun; it is also more probable than fission (involving uranium or plutonium) given relative abundances of the elements involved.

Energy Production Equation – I



Consider spherical shell of thickness dr in a spherically symmetric star of radius R in which energy transport is radial and time variations are unimportant.

$L(r)$ - rate of energy flow across sphere of radius r.

$L(r+dr)$ - rate of energy flow across sphere of radius $r+dr$.

Because the shell is thin

$$dV(r) = 4\pi r^2\, dr \quad \text{and} \quad dM(r) = 4\pi r^2\, \rho\, dr$$

Energy Production Equation – II

Define $\epsilon(r)$ as the energy release per unit mass in unit time and hence rate of energy release in the shell is $4\pi r^2 \rho(r)\,\epsilon(r)\,dr$. As energy is conserved

$$L(r+dr) = L(r) + 4\pi r^2 \rho(r)\,\epsilon(r)\,dr \qquad \text{and therefore}$$

$$\frac{L(r+dr) - L(r)}{dr} = 4\pi r^2 \rho(r)\,\epsilon(r)\,.$$

In the limit as $dr \to 0$

$$\frac{dL(r)}{dr} = 4\pi r^2 \rho(r)\,\epsilon(r)\,.$$

Here we have the Equation of Energy Production which is the **Third Equation of Stellar Structure**.

As the unknowns are $P(r)$, $M(r)$, $L(r)$, $\rho(r)$ and $\epsilon(r)$ energy transport needs to be revisited before stellar structure equations can be solved.

Equation of Radiation Transport – I

As previously shown, the radiation pressure may be expressed as

$$p = \frac{4\pi}{3c}\int_0^\infty B_\nu(T)\,d\nu = \frac{1}{3}\,a\,T^4$$

and differentiating with respect to T gives

$$\frac{dp}{dT} = \frac{4\pi}{3c}\int_0^\infty \frac{dB_\nu(T)}{dT}\,d\nu = \frac{4}{3}\,a\,T^3.$$

And so

$$\int_0^\infty \frac{dB_\nu(T)}{dT}\,d\nu = \frac{a\,c\,T^3}{\pi}.$$

Instead of using the absorption coefficient (κ_ν) in units of cm^{-1} or m^{-1}, an absorption coefficient ($\boldsymbol{\kappa_\nu}$) in units of $\mathrm{cm}^2/\mathrm{gm}$ or m^2/kg is adopted. Clearly

$$\kappa_\nu = \boldsymbol{\kappa_\nu}\,\rho.$$

A previously derived solution of the radiative transfer equation in the diffusion approximation is

$$H_\nu = -\frac{1}{3}\,\frac{1}{\boldsymbol{\kappa_\nu}\rho}\,\frac{dB_\nu}{dT}\,\frac{dT}{dr}$$

Equation of Radiation Transport – II

Integrating with respect to frequency gives

$$\int_0^\infty H_\nu \, d\nu = -\frac{1}{3\rho}\frac{dT}{dr}\int_0^\infty \frac{1}{\boldsymbol{\kappa_\nu}}\frac{dB_\nu}{dT}\,d\nu$$

$$= -\frac{1}{3\rho}\frac{dT}{dr}\frac{1}{\bar{\boldsymbol{\kappa}}_{\mathbf{Ross}}}\int_0^\infty \frac{dB_\nu}{dT}\,d\nu$$

$$= -\frac{1}{3\rho}\frac{dT}{dr}\frac{1}{\bar{\boldsymbol{\kappa}}_{\mathbf{Ross}}}\frac{a\,c\,T^3}{\pi}$$

$$= \frac{L(r)}{4\pi}\frac{1}{4\pi r^2}$$

where the definition of $\bar{\boldsymbol{\kappa}}_{\mathbf{Ross}}$ has been used and the resulting integral evaluated using the radiation pressure expression. Since $ac = 4\sigma$

$$\frac{dT}{dr} = \frac{3\rho\bar{\boldsymbol{\kappa}}_{\mathbf{Ross}}}{64\pi r^2\sigma T^3}L(r)$$

which gives the temperature gradient in a stellar envelope, in the absence of convection, at distance r from the stellar centre; it is the **Fourth Equation of Stellar Structure**.

Equation of Radiation Transport – III

As already explained, a stellar atmosphere or envelope becomes convective when the Schwarzschild Criterion

$$1 - \frac{1}{\gamma} \quad < \quad \left|\frac{d\ \ln T}{d\ \ln P}\right|_{\rm rad}$$

is satisfied. In other words, the adiabatic gradient becomes less than the radiative gradient. If the condition is satisfied, large scale rising and falling motions transport energy upwards.

The criterion can be satisfied in two ways:

- the ratio of specific heats (γ) is close to unity or
- the temperature gradient is very large.

For example, if a large amount of energy is released at the centre of a star, it may require a large temperature gradient to carry that energy away. Hence where nuclear energy is being released, convection may occur.

Alternatively, in the cool outer layers of a star, gas may only be partially ionised and much of the heat used to raise the temperature of the gas goes into ionisation; in these circumstances the specific heats at constant pressure and volume are almost the same and $\gamma \sim 1$.

Solution of Stellar Structure Equations – I

The four equations of stellar structure

$$\frac{dP(r)}{dr} = -\frac{G\,M(r)\,\rho(r)}{r^2}$$

$$\frac{dM(r)}{dr} = 4\,\pi\,r^2\,\rho(r)$$

$$\frac{dL(r)}{dr} = 4\,\pi\,r^2\,\rho(r)\,\epsilon(r)$$

$$\frac{dT(r)}{dr} = \frac{3\rho(r)\bar{\boldsymbol{\kappa}}_{\mathbf{Ross}}}{64\pi r^2 \sigma T(r)^3}\,L(r)$$

have been derived and the quantities $P(r)$, $\bar{\boldsymbol{\kappa}}_{\mathbf{Ross}}(r)$ and $\epsilon(r)$ are all also considered to be functions of ρ, T and chemical composition. The dependence of $P(r)$ on ρ, T and chemical composition is referred to as the **Equation of State**.

The obvious boundary conditions are:

- At the stellar centre, $r = 0$, $M(r) = 0$ and $L(r) = 0$.
- At the stellar surface $\rho \sim 10^{-4}$ kg m$^{-3} \ll \bar{\rho}_\odot \sim 1.4 \times 10^3$ kg m^{-3} and $T \sim T_{\rm eff} = 5780\,{\rm K} \ll \bar{T}_\odot \sim 2 \times 10^6$ K. At the surface $r = R$ and $\rho = T = 0$ is adopted.

Solution of Stellar Structure Equations – II

From a theoretical point of view, it is the mass of a star which is chosen, and while some mass is lost during evolution, it does not change by orders of magnitude as in the case of the radius. $M(r)$ is therefore a more natural choice of independent variable; the other three stellar structure equations are divided by the equation of mass conservation, which itself is inverted giving:

$$\frac{dP(r)}{dM(r)} = -\frac{G\,M(r)}{4\,\pi\,r^4},$$

$$\frac{dr}{dM(r)} = \frac{1}{4\,\pi\,r^2\,\rho(r)},$$

$$\frac{dL(r)}{dM(r)} = \epsilon(r) \quad \text{and}$$

$$\frac{dT(r)}{dM(r)} = \frac{3\bar{\boldsymbol{\kappa}}_{\mathbf{Ross}}\,L(r)}{16\,\pi^2\,r^4\,a\,c\,T(r)^3}.$$

The time taken by a star to consume all its nuclear energy is the nuclear time-scale given by

$$t_{\rm nuc} \sim \frac{\eta\,M\,c^2}{L}$$

where η is the mass-fraction converted into energy. For the Sun $t_{\rm nuc\odot} \sim 1.0 \times 10^{11}$ years. Moreover, for the Sun $t_{\rm d\odot} \sim 2300$ seconds and $t_{\rm th\odot} \sim 3 \times 10^7$ years. Clearly

$$t_{\rm d\odot} \ll t_{\rm th\odot} \ll t_{\rm nuc\odot}.$$

Solution of Stellar Structure Equations – III

Solution of the coupled differential equations of stellar structure allows the complete structure of a star, whose chemical composition and mass have been specified, to be determined. Limitations do need to be remembered:

- Evolution needs to proceed slowly when compared with $t_{\rm d}$ so that pulsations may be ignored; for the Sun this is certainly true.
- It has also been assumed that time-dependence can be omitted from the equation of energy generation if $t_{\rm nuc} \gg t_{\rm th}$; again, for the Sun this is certainly the case.
- If there are no bulk motions in the stellar interior, any changes in chemical composition are localised in the element of material in which nuclear reactions occurred. Chemical composition is then a function of mass, age and distance from the stellar centre.
- When bulk motions do occur, as with convection, stellar structure equations need to be supplemented by equations describing the rates of change of abundances of each chemical element. If $C_{\rm X,Y,Z}$ represents the chemical composition in terms of mass fractions for hydrogen (X), helium (Y) and metals (Z) at some mass-point $M(r)$ and time t_0 then

$$\frac{\partial [C_{\rm X,Y,Z}]_{M(r),t_0}}{\partial t} = f(\rho, T, C_{\rm X,Y,Z})$$

and at some later time $t_0 + \delta t$

$$[C_{\rm X,Y,Z}]_{M(r),(t_0+\delta t)} = [C_{\rm X,Y,Z}]_{M(r),t_0} + \frac{\partial [C_{\rm X,Y,Z}]_{M(r),t_0}}{\partial t}\,\delta t.$$

Solution of Stellar Structure Equations – IV

In convective regions, energy transport by convection needs to be included in the solution of stellar structure equations.

- As the temperature difference and velocity of rising elements are determined by the difference between the actual and adabatic temperature gradients, the former cannot be much larger than the latter.
- It is therefore a reasonable approximation to assume both gradients are equal in which case

$$\frac{P(r)}{T(r)}\frac{dT(r)}{dP(r)} = \frac{\gamma - 1}{\gamma}$$

- Stellar structure equations supplemented as described above are first solved for each concentric shell, into which the stellar interior has been discretised, assuming convection is unimportant in the first instance.
- The Schwarschild-Criterion is then applied to each concentric shell and the convective contribution to luminosity, computed by the Mixing Length Theory, included in the total shell luminosity if conditions permit convection to occur.
- Iteration then proceeds until a converged solution is attained for each concentric shell and the star as a whole.

Lecture 4: Summary

Essential points covered in fourth lecture:

- dynamical and thermal timescales, and the Virial Theorem;
- mean temperature and mean density of stellar envelope gas, and pressure at a stellar centre;
- the physical state of stellar material;
- energy generation in stars must be by fusion reactions;
- the significance of radiation pressure as a function of stellar mass and
- the equations of stellar structure.

It has been assumed that stellar structure can be determined assuming that changes due to stellar evolution are too slow to influence it; in general (though not always) this is a good approximation.

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