Introduction

Stars form as a consequence of the gravitational collapse of a gas cloud.

- Irrespective of mass, the collapsing gas cloud is fully convective and follows the Hayashi Line in the Hertzsprung-Russell Diagram.
- A fully convective star has to be on the Hayashi line.
- The region to the right of the Hayashi Line in the Hertzsprung-Russell Diagram is a "forbidden zone".
- All fully convective stars, such as very low mass M dwarfs, will remain on the Hayashi line throughout their Main Sequence evolution.
- Once on the Main Sequence, helium is produced by the PP-Chain for low mass stars and mostly by the CNO cycle in higher mass stars.
- Main Sequence lifetime falls off as roughly the inverse square of the stellar mass.

Stellar Evolution Tracks and Isochrones

The study of stellar evolution involves:

- Calculating some initial stellar model by the simultaneous solution the four equations of stellar structure along with the three supplementary equations.
- Identify nuclear reactions and other processes such as convection and gravitational contraction which can change the structure of the model just computed.
- For an appropriate time-step, estimate a new stellar structure and use this as a starting approximation for a new solution of the stellar structure and supplementary equations.
- Repeating the process yields an evolutionary track in the Hertzsprung-Russell Diagram (HRD) ($\log_{10} L$ vs. $\log_{10} T$) which shows how the HRD position of a star of given mass and initial chemical composition changes with time.
- Evolution tracks calculated for a range of stellar masses allows the dependence of evolution on mass to to be studied.
- Points identifying a fixed age on evolution tracks for different masses define an isochrone.
- Isochrones may be extracted for any number of ages and compared with observed colour-magnitude diagrams for globular clusters to determine their ages.

Five Sections of the Hertzsprung-Russell Diagram

All stars of sufficient mass have a core hydrogen-burning phase which is the Main Sequence, but subsequent evolution depends on mass:

- Brown Dwarfs (and Planets) have masses below $0.08M_\odot$ or 80 M_{Jup} which is the estimated lower mass limit for T_c to reach the 1.5×10^7 K needed for core hydrogen burning.
- Red Dwarfs are stars whose main-sequence lifetime exceeds the present age of the Universe estimated as 1.37×10^{10} years. Models yield an upper mass limit ($0.7M_\odot$ for stars that must still be on the Main Sequence, even if they are as old as the Universe.
- Low-Mass Stars ($0.7 \leqslant M \leqslant 2M_\odot$) in most cases end up losing most of their mass to become planetary nebulae before evolving on to the white dwarf cooling track.
- Intermediate-Mass Stars ($2 \leqslant M \leqslant 8M_\odot$) have evolution tracks similar to those of low-mass stars, but always at higher luminosity.
- High-Mass (or massive) stars ($M > 8M_\odot$) have distinctly different lifetimes and evolutionary paths.

Star Formation – I

A rough estimate of the radius of a protostar after the dynamical collapse can be obtained by assuming all gravitational energy released during the collapse is absorbed in the dissociation of molecular hydrogen (requiring $\chi_{\mathrm{H_2}} = 4.5\,\mathrm{eV}$ per $\mathrm{H_2}$ molecule) and ionisation of hydrogen $\chi_{\mathrm{H}} = 13.6\,\mathrm{eV}$ and helium $\chi_{\mathrm{He}} = 79\,\mathrm{eV}$. Assuming the collapse starts with the initial gas cloud having an infinite radius, the protostar radius R_{p} is given by

$$\frac{G\,M^2}{2\,R_{\mathrm{p}}} \simeq \frac{M}{m_{\mathrm{H}}}\left(\frac{X}{2}\chi_{\mathrm{H_2}} + \chi_{\mathrm{H}} + \frac{Y}{4}\chi_{\mathrm{He}}\right)$$

where the left hand side follows from the Virial Theorem as already discussed. Taking $X \simeq 0.7$ and $Y = 1 - X$ gives

$$\frac{R_{\mathrm{p}}}{R_{\odot}} = 60\,\frac{M}{M_{\odot}}.$$

Since the protostar is in hydrostatic equilibrium to a good approximation, an earlier application of the Virial Theorem may be followed to give the mean temperature in its stellar envelope as

$$\begin{aligned}\bar{T} &\gtrsim \frac{G\,M\,m_{\mathrm{H}}\bar{\mu}}{6\,k\,R_{\mathrm{p}}} \\ &\simeq 6 \times 10^4\ \mathrm{K}\end{aligned}$$

and which is independent of mass.

Star Formation – II

At these low temperatures the opacity is very high, rendering radiative transport inefficient and making the protostar convective throughout. Protostar properties are then:

- If $T_{\rm eff}$ is small enough, stars become completely convective.
- In that case energy transport is very efficient through out the interior of a star, and a tiny superadiabaticity $[(d\ln T/d\ln P) - (d\ln T/d\ln P)_{\rm ad}]$ is sufficient to transport a very large energy flux.
- The structure of such a star can be said to be adiabatic.
- Since an almost arbitrarily high energy flux can be carried, the luminosity of a fully convective star is practically independent of its structure; this is in marked contrast to a star in radiative equilibrium, for which the luminosity is strongly linked to the temperature gradient.

For a fully convective polytropic star ($n = 3/2$) with H^- opacity, it can be shown that $T_{\rm eff}$ is related to the total mass (M), luminosity (L) and mean molecular weight $\bar{\mu}$ by

$$T_{\rm eff} \sim 2600 \bar{\mu}^{13/51} \left(\frac{M}{M_\odot}\right)^{7/51} \left(\frac{L}{L_\odot}\right)^{1/102} \text{K}.$$

Note that $T_{\rm eff}$ is nearly independent of L and only weakly dependent on M and $\bar{\mu}$. All fully convective stars will therefore fall in more or less the same cool area of the HRD.

Star Formation – III

It therefore follows that:

- Fully convective stars of a given mass occupy an almost vertical line in the HRD (*i.e.* with $T_{\rm eff} \simeq$ constant).
- The line is known as the Hayashi line.
- The region to the right of the Hayashi line in the HRD is a *forbidden region* for stars in hydrostatic equilibrium.
- Stars to the left of the Hayashi line (at higher $T_{\rm eff}$) cannot be fully convective but must have some portion of their interior in radiative equilibrium.

The forbidden region of the HRD exists because any star which evolves into it must have a superadiabatic gradient (in addition to the outer layers which are always superadiabatic). A very large convective energy flux results, far exceeding normal stellar luminosities. Energy is very rapidly (on a dynamical timescale) transported outwards, reducing the temperature gradient in the superadiabatic region until it is adiabatic once more and the star is again on the Hayashi line.

Pre-Main Sequence Contraction – I

A newly formed star emerges from dynamical collapse and settles on the Hayashi line appropriate for its mass; it is now a pre-main sequence (PMS) star having a radius roughly as given above.

- The temperature is not high enough for nuclear burning and the energy source for its luminosity is gravitational contraction.
- As dictated by the Virial Theorem, the PMS star internal temperature increases as a consequence.
- The PMS star remains fully convective for as long as the opacity remains high; it contracts along its Hayashi line and thus its luminosity decreases.
- Since fully convective stars are accurately described by $n = 1.5$ polytropes, this phase of contraction is homologous to a very high degree.
- Thus the central temperature increases as $T_c \propto {\rho_c}^{1/3} \propto R^{-1}$.
- As the internal temperature increases, opacity will decrease and the point will be reached where the Schwarzschild Criterion for convective instability are no longer satisfied.
- A radiative core then develops and the PMS star moves to the left in the HRD, away from the Hayashi line and towards higher $T_{\rm eff}$.

Pre-Main Sequence Contraction – II

- As the radiative core grows, the extent of the convective envelope decreases.
- Contraction continues but luminosity is now roughly constant as the effective temperature increases.

Before thermal equilibrium on the Zero-Age Main Sequence (ZAMS) is reached, several nuclear reactions have already set in.

- Deuterium is a very fragile nucleus that reacts readily with protons (second reaction in the PPI Chain) and all deuterium in the stellar core is removed at $T \gtrsim 1.0 \times 10^6$ K.
- A similar but much smaller effect happens later (at higher T) when the initially present lithium, with mass fraction $\lesssim 10^{-8}$, is depleted.

$$
\begin{array}{ll}
1) & {}^{1}\mathrm{H} + {}^{6}\mathrm{Li} \rightarrow {}^{7}\mathrm{Be} \\
2) & {}^{7}\mathrm{Be} + \mathrm{e}^{-} \rightarrow {}^{7}\mathrm{Li} + \nu \\
3) & {}^{1}\mathrm{H} + {}^{7}\mathrm{Li} \rightarrow {}^{8}\mathrm{Be} \\
4) & {}^{8}\mathrm{Be} \rightarrow {}^{4}\mathrm{He} + {}^{4}\mathrm{He}
\end{array}
$$

Pre-Main Sequence Contraction – III

- In addition, the first CNO-Cycle reaction $^{12}\mathrm{C}(^{1}\mathrm{H},\gamma)^{13}\mathrm{N}$ is already activated at a temperature below that of the full CNO-Cycle, due to the relatively large $^{12}\mathrm{C}$ abundance compared with the equilibrium CNO abundances.
- The next two CNO-Cycle reactions ensure almost all $^{12}\mathrm{C}$ in the core is converted to $^{14}\mathrm{N}$ before the ZAMS is reached.
- Time taken by a PMS star to reach ZAMS is basically the Kelvin-Helmholtz time scale previously derived to be
$$t_{\mathrm{th}} = \frac{G\,M^2}{L\,R},$$
a time scale dominated by the final stages of contraction when L and R are comparitively small.
- A crude average of the ZAMS homology relations already obtained for "low" and "high" mass stars suggest $L\,R \sim M^{4.5}$ and so the PMS lifetime is
$$\tau_{\mathrm{PMS}} \sim 3 \times 10^7 (M/M_\odot)^{-2.5}\ \mathrm{yr}.$$
Thus massive protostars reach the ZAMS much earlier than lower mass stars.

Main Sequence Star Characteristics – I

Main Sequence stars obey several relations:

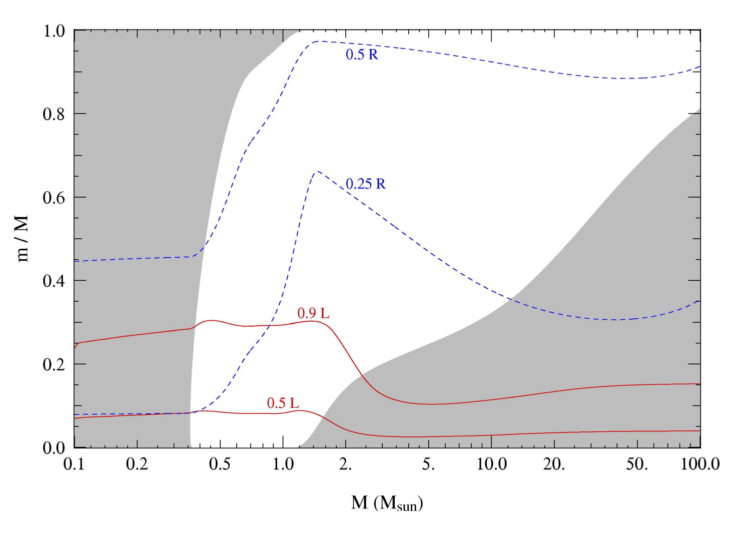
- As already shown by homology, $L \propto M^{a_5}$ where for low-mass and high-mass stars $a_5 = 5.5$ and $a_5 = 3.0$ were deduced respectively. The flattening at higher masses is due to the increased contribution of radiation pressure in the central core, which helps support the star and decreases the central temperature slightly. A flattening also occurs at the very low masses due to the increasing importance of convection for stellar structure.

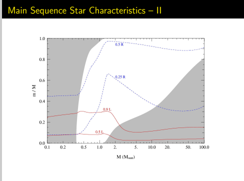
- Homology also indicates that there is a mass-radius relation $R \propto M^{a_1}$ but there is significant break at $M \sim 1\,M_\odot$ due to the onset of convection in the envelope. Convection increases the energy flow out of a star which causes it to contract slightly. Stars with convective envelopes therefore lie below the mass-radius relation and above the mass-luminosity relation for non-convective stars.

- Uncertainties in the mixing-length scale (α) affect the computed stellar R and $T_{\rm eff}$ slightly. For example, a factor of two increase in α translates to a change of $+0.03$ in $\log T_{\rm eff}$.

- The depth of the convective envelope in terms of $M_{\rm env}/M$ increases with decreasing mass. Stars with $M \sim 1\,M_\odot$ have extremely thin convective envelopes, while those with $M \lesssim 0.3\,M_\odot$ are entirely convective. Nuclear burning ceases at $M \simeq 0.08\,M_\odot$.

Main Sequence Star Characteristics – II



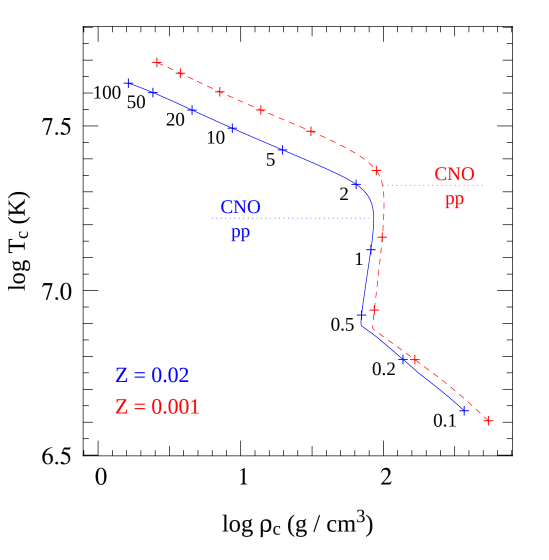


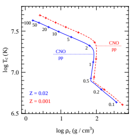
The diagram shows the occurence of convective regions (gray shading) on the Zero Age Main Sequence (ZAMS) in terms of the fractional mass coordinate m/M as a function of the stellar mass M, for detailed stellar models with a composition $X = 0.70, Z = 0.02$. The solid (red) lines show the mass shells inside which 50% and 90% of the total luminosity is produced. The dashed (blue) lines show the mass coordinate where the radius r is 25% and 50% of the stellar radius R.

Main Sequence Star Characteristics – III

- Main sequence star interiors are extremely hot ($T > 10^6\,\mathrm{K}$) and the fall-off to surface temperatures ($T \sim 10^4\,\mathrm{K}$) takes place in a very thin region near the surface.
- Nuclear energy generation is restricted to a very small $M(r)$ near the centre of the star. The rapid fall-off of $\epsilon(r)$ with r reflects the extreme sensitivity of energy generation to temperature.
- Main sequence stars with $M \lesssim 1\,M_\odot$ generate most of their energy through the PP-Chain; at greater masses the CNO-Cycle is responsible for most of the energy generation. The changeover is responsible for a shift in the homology relations in stellar interiors.
- CNO burning exhibits an extreme temperature dependence and consequently those stars for which the CNO-Cycle is the most important source of energy have very large values of $L(r)/(4\,\pi\,r^2)$ in the core. The resulting high temperature gradient ensures efficient convective energy transport in the core.
- Because of the extreme sensitivity of CNO burning to temperature, nuclear reactions in stars having $M \gtrsim 1\,M_\odot$ are confined to a small region which is very much smaller than the convective core.
- As the stellar mass increases, so does the size of the convective core; this is a direct consequence of the large increase in $\epsilon(r)$ with temperature. Supermassive stars with $M \sim 100\,M_\odot$ would be entirely convective.

Main Sequence Star Characteristics – IV





Central temperature versus central density for detailed Zero Age Main Sequence (ZAMS) models with $X = 0.70, Z = 0.02$ (blue solid line) and with $X = 0.757, Z = 0.001$ (red dashed line). Plus symbols indicate models for specific masses (in units of $M_\odot$). The dotted lines indicate the approximate temperature border between energy production dominated by the CNO-Cycle and the PP-Chain. This gives rise to a change in slope of the $T_\mathrm{c}, \rho_\mathrm{c}$ relation.

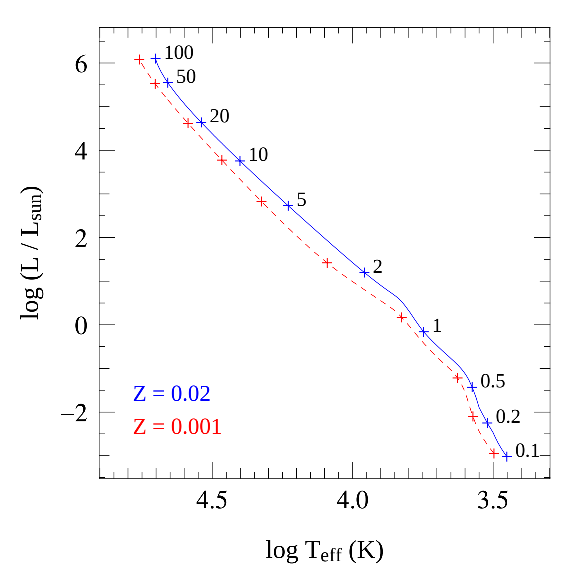
Main Sequence Star Characteristics – V

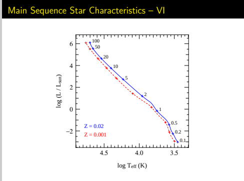
- Position on the ZAMS depends on both stellar mass and initial helium abundance. The higher $\bar{\mu}$ implied by a higher helium abundance translates to a lower core pressure. Helium-rich stars are therefore more condensed which (through the Virial Theorem) mean they have higher core temperatures, nuclear reaction rates and luminosities.

- Changes in metallicity shift the location of the ZAMS. A lower metallicity reduces bound-free absorption throughout the star; this allows the energy to escape more easily as may be seen from the equation of radiation transport used in stellar interiors
$$L = -\frac{16\,\pi\,a\,c\,r^2\,T^3}{3\,\kappa\,\rho}\,\frac{dT}{dr}.$$
The effect is slightly less important for higher mass stars since the energy generation rate is proportional to Z in CNO burning, but the smaller κ remains the more important factor. The luminosity increase also results in a hotter surface temperature since $L \propto R^2\,{T_{\rm eff}}^4$.

- Changes in the metallicity and helium abundance also affect the precise mass at which a convective core disappears. A greater helium abundance forces an increase in $\rho\,T$ to maintain the pressure. The result is an increased nuclear reaction rate, luminosity, temperature gradient and probability of convection. Similarly, a decrease in Z also results in an increased nuclear reaction rate, luminosity, temperature gradient and probability of convection.

Main Sequence Star Characteristics – VI





The location of the Zero-Age Main Sequence (ZAMS) in the Hertzsprung-Russell diagram for homogeneous, detailed stellar models with $X = 0.7, Z = 0.02$ (blue solid line) and with $X = 0.757, Z = 0.001$ (red dashed line). Plus symbols indicate models for specific masses (in units of $M_\odot$). ZAMS models for metal-poor stars are hotter and have smaller radii. Relatively low-mass stars at low metallicity are also more luminous than their metal-rich counterparts.

Main Sequence Star Characteristics – VII

Since main sequence stars are in virial equilibrium

$$T_{\rm c} \propto \frac{M}{R} \quad \text{and so} \quad {T_{\rm c}}^3 \propto \left(\frac{M}{R}\right)^3 \propto \rho_{\rm c}\, M^2.$$

And from the energy generation equation

$$\frac{dL(r)}{dM(r)} = \epsilon(r) \propto \rho_{\rm c}\, T^{\eta} \quad \text{and therefore} \quad L \propto \rho_{\rm c}\, T^{\eta}\, M.$$

Now the mass-luminosity relation for a homologous star is $L \propto M^{a_5}$. Eliminating L and substituting for $T_{\rm c}$ gives

$$M^{a_5} \propto \rho_{\rm c} \left({\rho_{\rm c}}^{1/3}\, M^{2/3}\right)^{\eta} M$$

and consequently

$$\rho_{\rm c} \propto M^{(3a_5-3-2\eta)/(3+\eta)}.$$

There are three cases to consider

- $M \gtrsim 2\, M_\odot$ with $a_5 = 3$ and $\eta = 18 \rightarrow (3a_5 - 3 - 2\eta)/(3+\eta) = -1.43$,
- $0.7 \lesssim M \lesssim 2\, M_\odot$ with $a_5 = 5.5$ and $\eta = 4 \rightarrow (3a_5 - 3 - 2\eta)/(3+\eta) = +0.79$ and
- $M \lesssim 0.7\, M_\odot$ with $a_5 = 3$ and $\eta = 4 \rightarrow (3a_5 - 3 - 2\eta)/(3+\eta) = -0.29$.

The very low mass case above has not been derived as convection had been ignored in the earlier discussion on homologous stars.

Main Sequence Star Characteristics – VIII

- At large and small stellar masses, the central density ρ_c follows its homology relation, but becomes almost independent of mass near $\sim 1\,M_\odot$.
- The behaviour can be understood from the change in sign of $(3a_5 - 3 - 2\eta)/(3+\eta)$ when $0.7 \lesssim M \lesssim 2.0\,M_\odot$ as demonstrated on the previous slide.
- ZAMS stars in the mass range $0.7 \lesssim M \lesssim 2.0\,M_\odot$ are distinguished from lower and higher mass stars in that they are convective only in the outermost layers, which will not significantly affect the central density.

Luminosity Evolution – I

As a star on the Main Sequence burns hydrogen in its core, its luminosity will increase; this is due to the star's higher mean molecular weight. Since the core has fewer particles to support it, ρT must increase, and so must the rate of nuclear reactions.

By the Virial Theorem

$$\frac{1}{\bar{\mu}} T \propto \frac{M}{R} \qquad \text{or} \qquad T \propto \bar{\mu}\, M^{2/3} \rho^{1/3}$$

since $R \propto (M/\rho)^{1/3}$. Assuming that energy is transported radiatively, then the equation of radiation transport as used in stellar interiors gives

$$L = -\frac{16\, a\, c\, r^2\, T^3}{3\, \kappa\, \rho} \frac{dT}{dr}.$$

Very roughly,

$$\frac{dT}{dr} \sim \frac{T - T_0}{0 - R} \simeq -\frac{T}{R} \qquad \text{and so} \qquad L \propto \frac{R\, T^4}{\kappa\, \rho}.$$

For main sequence stars, Kramer's law opacity dominates except in the centre, where electron scattering dominates, thus

$$L \propto \frac{R\, T^{15/2}}{\rho^2}.$$

Luminosity Evolution – II

Using proportionalities for R and T above, the proportionality for L is

$$L \propto \frac{M^{1/3}\, T^{15/2}}{\rho^{7/3}} \propto \frac{M^{1/3}\, \bar{\mu}^{15/2}\, M^5 \rho^{5/2}}{\rho^{7/3}} \propto M^{16/3}\, \rho^{1/6}\, \bar{\mu}^{15/2}.$$

Since the stellar mass (M) is essentially constant, and the density dependence is very weak, the above equation implies:

$$\frac{L(t)}{L(0)} = \left[\frac{\bar{\mu}(t)}{\bar{\mu}(0)}\right]^{\psi}$$

where t is the elapsed time (or age) since the star was on the ZAMS and $\psi = 15/2$.

The previously derived expression for the mean molecular weight was

$$\bar{\mu} = \frac{4}{6X + Y + 2} \qquad \text{which becomes} \qquad \bar{\mu} = \frac{4}{3 + 5X}$$

if metal content is ignored so that $X + Y = 1$.

Finally, if the fusion process releases energy Q per unit mass of hydrogen fused, the rate of hydrogen consumption is related to luminosity by

$$\frac{dX}{dt} \simeq -\frac{L}{M\,Q}.$$

Luminosity Evolution – III

The rate of change of luminosity of a star evolving on the Main Sequence may be expressed as:

$$\frac{dL(t)}{dt} = \left(\frac{dL(t)}{d\bar{\mu}(t)}\right)\left(\frac{d\bar{\mu}(t)}{dX(t)}\right)\left(\frac{dX(t)}{dt}\right) \qquad \text{where}$$

$$\frac{dL(t)}{d\bar{\mu}(t)} = \psi\left(\frac{L(0)}{\bar{\mu}(0)}\right)\left(\frac{\bar{\mu}(t)}{\bar{\mu}(0)}\right)^{\psi-1} \qquad \text{and}$$

$$\frac{d\bar{\mu}(t)}{dX(t)} = -\,5\,\frac{4}{(3+5X(t))^2} = -\,\frac{5}{4}\,\bar{\mu}(t)^2.$$

After substitution

$$\frac{dL(t)}{dt} = \psi\left[\frac{L(0)}{\bar{\mu}(0)}\left(\frac{\bar{\mu}(t)}{\bar{\mu}(0)}\right)^{\psi-1}\right]\left(-\,\frac{5}{4}\,\bar{\mu}(t)^2\right)\left(-\,\frac{L(t)}{M\,Q(t)}\right),$$

$$= \frac{5}{4}\,\psi\,L(0)\,\bar{\mu}(0)\left(\frac{\bar{\mu}(t)}{\bar{\mu}(0)}\right)^{\psi+1}\frac{L(t)}{M\,Q(t)} \qquad \text{and}$$

$$= \frac{5}{4}\,\psi\,L(0)\,\bar{\mu}(0)\,\frac{L(t)}{M\,Q(t)}\left(\frac{L(t)}{L(0)}\right)^{(\psi+1)/\psi}$$

Luminosity Evolution – IV

Rearranging leads to the differential equation

$$L(t)^{(-2-1/\psi)}\,dL(t) = \frac{5\,\psi\,\bar{\mu}(0)}{4\,L(0)^{1/\psi}\,M\,Q(t)}\,dt$$

which on integrating leads to

$$L(t)^{-(\psi+1)/\psi} - L(0)^{-(\psi+1)/\psi} = \frac{5\,(\psi+1)\,\bar{\mu}(0)}{4\,L(0)^{1/\psi}\,M\,Q(t)}\,t \qquad \text{or}$$

$$L(t) = L(0)\,\left[1 - \frac{5}{4}(\psi+1)\frac{\bar{\mu}(0)\,L(0)}{M\,Q}t\right]^{-\psi/(\psi+1)} .$$

If $\bar{\mu}(0) \simeq 0.6$, $\psi = 15/2$, $L(0) = 4 \times 10^{26}$ Watts, $M = 2.0 \times 10^{30}$ kg and $Q = 6.0 \times 10^{14}$ J kg^{-1} then in solar units

$$\frac{L(t)}{L_\odot} = \frac{L(0)}{L_\odot}\left[1 - 0.3\,\frac{L(0)}{L_\odot}\,\frac{t}{t_\odot}\right]^{-15/17}$$

For the Sun, the above relation corresponds to $L_\odot = 1.26\,L(0)$ which compares with detailed models suggesting the Sun has brightened since the ZAMS by a factor of 1.37. In other words, as a ZAMS star, the Sun was $\sim 25\%$ less luminous than it is now.

Convective Regions

As already discussed, three types of ZAMS star may be distinguished by their convective and radiative regions:

- for $M < 0.35\,M_\odot$ the star is completely convective,
- for $0.35\,M_\odot < M < 1.2\,M_\odot$ the core is radiative but the envelope is convective and
- for $M > 1.2\,M_\odot$ the core is convective but the envelope is radiative.

The behaviour may be understood from the Schwarzschild Criterion which are that convection occurs when $\nabla_{\rm rad} > \nabla_{\rm ad}$, having written $\nabla \equiv d\ln T/d\ln P$.

- The CNO-Cycle, which dominates energy production when $M > 1.2\,M_\odot$ has a very high burning rate temperature sensitivity ($\eta = 18$), implying a rapid decrease in ϵ with falling temperature away from the stellar centre. $\nabla_{\rm rad}$ is therefore large at the stellar centre and energy transport in the core is convective.
- When $M < 1.2\,M_\odot$, energy generation is dominated by the PP-Chain ($\eta = 4$) which has a much smaller burning rate temperature sensitivity. Burning is therefore distributed over a larger volume, $\nabla_{\rm rad}$ is too small for convection and the core remains radiative.
- With decreasing M, $T(r)$ also decreases leading to a higher opacity and $\nabla_{\rm rad}$; the envelope therefore becomes convective. The depth to which convection penetrates in the envelope increases as M decreases and for $M < 0.35\,M_\odot$, the entire star is convective.

Stars Powered by the CNO-Cycle

Main Sequence stars powered by the CNO-Cycle ($M \gtrsim 1.2\,M_\odot$) have an evolution on it which distinguish them from stars powered by the PP-Chain:

- As the conversion of H $\rightarrow$ He raises $\bar{\mu}$, the perfect gas law requires that P_c must decrease because observations show that L does not increase and therefore T_c is constant ($\epsilon_\mathrm{CNO} \propto \rho T^{18}$); this can only happen if the pressure exerted on the core by the envelope decreases. In other words, the star must expand in order to maintain hydrostatic equilibrium.
- CNO-Cycle temperature sensitivity ensures energy production is concentrated at the stellar centre; the high temperature gradient gives efficient convection which mixes the H and He throughout the convective core. H available for burning is therefore increased and this increases the Main Sequence lifetime.
- Towards the end of the Main Sequence, X becomes very small in the convective core and T has to increase to maintain energy production. Once $X \simeq 0$ throught the convective core, burning will cease the star radiates more energy than is generated and will undergo an overall contraction. This defines the end of the Main Sequence evolution phase.
- Contraction results in a heating of the core and surrounding envelope to the point where the CNO-Cycle is ignited in a shell around the helium core.

Stars Powered by the PP-Chain

Main Sequence stars powered by the PP-Chain ($M \lesssim 1.2\,M_\odot$) have an evolution on it which distinguish them from stars powered by the CNO-Cycle:

- The lower temperature sensitivity ($\epsilon_{\rm PP} \propto \rho T^4$) means that $T_{\rm c}$ and $\rho_{\rm c}$ increase more than was the case for the CNO-Cycle. Therefore the outer layers need to expand less in order to maintain hydrostatic equilibrium in the core, and evolution is almost parallel to the ZAMS in the HRD.
- The lower temperature gradient in the core means that these are radiative. An H abundance gradient builds up in the core, with X lowest at the stellar centre. As a result, H is depleted gradually in the core and there is a smooth transition to H-shell burning.
- At solar metallicity and $1.1 \lesssim M \lesssim 1.2\,M_\odot$, $T_{\rm c}$ becomes high enough for the CNO-Cycle to take over from the PP-Chain; these stars therefore develop convective cores and end their lives on the Main Sequence in the same way as stars having $M \gtrsim 1.2\,M_\odot$.

Lifetime on the Main Sequence – I

The time (τ_{MS}) that a star spends on the Main Sequence is essentially the nuclear timescale previously discussed. Essentially the same result follows from the relation

$$\frac{dX}{dt} = -\frac{4\,m_{\rm H}}{Q_{\rm H}}\,\epsilon$$

where $Q_{\rm H}$ is the effective energy release of the reaction chain

$$4\,{}^{1}\mathbf{H} \to {}^{4}\mathrm{He} + 2\,\mathrm{e}^{+} + 2\,\nu_{\rm e}.$$

For a star in thermal equilibrium, integrating over all mass shells, gives

$$\frac{d\,M_{\rm H}}{dt} = -\frac{4\,m_{\rm H}}{Q_{\rm H}}\,L$$

where $M_{\rm H}$ is the total mass of hydrogen in the star. Integrating over the Main Sequence lifetime results in

$$\Delta M_{\rm H} = \frac{4\,m_{\rm H}}{Q_{\rm H}}\int_0^{\tau_{MS}} L\,dt = \frac{4\,m_{\rm H}}{Q_{\rm H}}\bar{L}\,\tau_{MS}$$

where $\bar{L}$ is the average luminosity over the Main Sequence lifetime.

Lifetime on the Main Sequence – II

Writing $\Delta M_{\rm H} = f_{\rm nuc}\, M$ and $f_{\rm nuc}$ as the product of the initial H mass-fraction X_0 and an effective core mass-fraction $q_{\rm c}$, inside which all H is consumed, gives

$$\tau_{\rm MS} = X_0\, q_{\rm c}\, \frac{Q_{\rm H}}{4\, m_{\rm H}}\, \frac{M}{\bar{L}}.$$

Since the variation of L during $\tau_{\rm MS}$ is modest and L increases dramatically with increasing M, the ZAMS relation $\bar{L} \propto M^{a_5}$ may be adopted, with $a_5 \sim 3.8$ on average. Therefore

$$\tau_{\rm MS} \propto M^{1-a_5}$$

and $\tau_{\rm MS}$ decreases dramatically with increasing mass, a trend which has important consequences for observed star cluster HRDs:

- All stars in a cluster are assumed to have been formed at the same time and therefore have the same age $\tau_{\rm cl}$
- Cluster stars with M above some limit $M_{\rm to}$ have $\tau_{\rm MS} < \tau_{\rm cl}$ and have therefore already left the Main Sequence.
- Similarly stars having $\tau_{\rm MS} > \tau_{\rm cl}$ are still on the Main Sequence and determining the mass at which stars turn-off ($M_{\rm to}$) gives an estimate of the cluster age.

Lifetime on the Main Sequence – III

The actual τ_{MS} depends on a number of other factors:

- The effective energy release $Q_{\rm H}$ depends on exactly which reactions are involved in energy production, and therefore has a small mass-dependence.
- More importantly, the exact value of $q_{\rm c}$ is determined by the H-profile left at the end of the Main Sequence; this is somewhat mass-dependent, especially for massive stars in which the relative size of the convective core tends to increase with mass.
- A larger convective core mass means a larger fuel reservoir and a longer τ_{MS}.
- Our poor understanding of convection and mixing in stars introduces considerable uncertainty in the size of the reservoir and therefore both in τ_{MS} for a particular mass, and the star's further evolution.

Convective Overshooting – I

The size of a convective region inside a star is expected to be larger than that defined by the Schwarzschild criterion because of convective overshooting; however, this is not reliably known from theory.

- In stellar evolution calculations, overshooting is usually parameterised as $d_{\rm ov} = \alpha_{\rm ov}\, H$ where H is the local pressure scale-height already introduced in the Mixing Length Theory.
- Other physical effects such as rotation may also mix material beyond the formal convective core boundary.
- Stellar evolution models which include overshooting generally provide a better match to observations; it is therefore understood to be significant in stars with sizeable convective cores on the Main Sequence.

Convective Overshooting – II

Overshooting has several important consequences for the evolution of a star:

- τ_{MS} is longer because of the larger H reservoir available.
- The increase in L and R during the Main Sequence evolution are greater, because of the larger region inside which $\bar{\mu}$ increases.
- The hydrogen-exhausted core mass is larger at the end of Main Sequence evolution which leads to
  - higher luminosities during evolution phases which follow the Main Sequence and
  - shorter lifetimes of post-main sequence phases.

Stellar evolution models computed with different values of $\alpha_{\rm ov}$ may be compared with the observed width of the Main Sequence in star clusters, and to the luminosities of evolved stars in binary systems. It is then possible to calibrate $\alpha_{\rm ov}$ if the luminosity difference between binary components or cluster $M_{\rm to}$ are well determined observationally. Tests suggest $\alpha_{\rm ov} \simeq 0.25$ is appropriate for $1.5 \lesssim M \lesssim 8.0 M_{\odot}$. For larger masses, $\alpha_{\rm ov}$ is poorly constrained.

Semi-Convection

Outside a convective core, a composition gradient develops.

- In such a region, an overstable oscillation pattern can develop on a thermal timescale.
- The region is slowly mixed and the composition gradient becomes smoothed out.
- The process, known as semi-convection, has an uncertain outcome and efficiency.
- Semi-convection is encountered during various phases of stellar evolution, most importantly during central H-burning in stars with $M \gtrsim 10\,M_\odot$ and during helium burning in low and intermediate mass stars.

Summary

Stars form as a consequence of the gravitational collapse of a gas cloud.

- Irrespective of mass, the collapsing gas cloud is fully convective and follows the Hayashi Line in the Hertzsprung-Russell Diagram.
- A fully convective star has to be on the Hayashi line.
- The region to the right of the Hayashi Line in the Hertzsprung-Russell Diagram is a "forbidden zone".
- All fully convective stars, such as very low mass M dwarfs, will remain on the Hayashi line throughout their Main Sequence evolution.
- Once on the Main Sequence, helium is produced by the PP-Chain for low mass stars and mostly by the CNO cycle in higher mass stars.
- Main Sequence lifetime falls off as roughly the inverse square of the stellar mass.

Acknowledgement

Material presented in this lecture on Main Sequence star evolution is based almost entirely on slides prepared by N. Langer (Universität Bonn) and by others for use at Pennsylvania State University.