Introduction

In considering evolution of stars after the Main Sequence, it is useful to make a division based on mass:

- Low-mass stars ($0.8 \lesssim M \lesssim 2.0 M_\odot$) are those that develop a degenerate He-core after Main Sequence evolution, leading to a relatively long-lived Red Giant Branch phase and to the ignition of He in a so-called helium flash.
- Intermediate-mass stars ($2.0 \lesssim M \lesssim 8.0 M_\odot$) stably ignite He in a non-degenerate core, but after central He burning develop a degenerate C/O core. Both low-mass and intermediate-mass stars shed their envelopes by a strong stellar wind, their remnants being C/O white dwarfs.
- Massive stars ($M \gtrsim 8 M_\odot$) ignite carbon in a non-degenerate core. Heavier elements are also ignited in the core until a Fe-core is formed and which subsequently collapses.

Schönberg-Chandrasekhar Limit – I

A star evolving off the Main Sequence will roughly follow the predictions of homology in the first instance:

- Specifically, L will increase due to the increase in the star's $\bar{\mu}$ while R remains roughly constant.
- The increase in L continues until $X < 0.05$ in the core.
- Once H is exhausted in the core, the star's structure must change dramatically (non-homologously).

The above can be understood by considering a star whose envelope is still homologous but whose core has no nuclear burning. If the core is not producing any luminosity, then to be in near thermal equilibrium it must be approximately isothermal; otherwise, energy would diffuse outward. The maximum surface pressure an isothermal core of temperature T_c can withstand is obtained from the Equation of Hydrostatic Equilibrium and the Virial Theorem.

Schönberg-Chandrasekhar Limit – II

Multiplying the Equation of Hydrostatic Equilibrium

$$\frac{dP}{dr} = -\frac{G\,M(r)}{r^2}\,\rho(r)$$

by $4\,\pi r^3\,dr$, and integrating by parts over the core, gives

$$\left[4\,\pi r^3 P(r)\right]_0^{R_\mathrm{c}} - \int_0^{R_\mathrm{c}} 12\,\pi r^2\,P(r)\,dr = -\int_0^{R_\mathrm{c}} \frac{G\,M(r)\,\rho(r)}{r^2}\,4\,\pi\,r^3\,dr,$$

which becomes

$$4\,\pi R_\mathrm{c}{}^3 P(R_\mathrm{c}) - \int_0^{M_\mathrm{c}} 3\,\frac{P(r)}{\rho(r)}\,dM(r) = -\int_0^{M_\mathrm{c}} \frac{G\,M(r)}{r}\,dM(r)$$

since

$$dM(r) = 4\,\pi\,r^2\,\rho(r)\,dr.$$

If the equation of state at the surface of the core is close to that of an ideal gas, then $P(r)/\rho(r) \propto T_\mathrm{c}$ and for an isothermal core T_c is constant and therefore

$$\int_0^{M_\mathrm{c}} 3\,\frac{P(r)}{\rho(r)}\,dM(r) = 3\,T_\mathrm{c}\,M_\mathrm{c}.$$

Schönberg-Chandrasekhar Limit – III

For all points inside the core, $r < R_c$ and so by the Virial Theorem

$$\int_0^{M_c} \frac{G\,M(r)}{r}\,dM(r) > \int_0^{M_c} \frac{G\,M(r)}{R_c}\,dM(r) = \frac{G\,{M_c}^2}{2\,R_c}.$$

The gas pressure at the surface of the core may therefore be written as

$$P(R_c) \;=\; K_1\,\frac{M_c\,T_c}{{R_c}^3} - K_2\,\frac{{M_c}^2}{{R_c}^4}$$

where $K_1 = 3/(4\,\pi)$ and $K_2 = G/(8\,\pi)$.

The maximum surface pressure for a core of radius R_c and mass M_c is then obtained from

$$\frac{dP(R_c)}{dR_c} = -3\,K_1\,\frac{M_c T_c}{{R_c}^4} + 4\,K_2\,\frac{{M_c}^2}{{R_c}^5} = 0$$

which gives

$$R_c < K_3\,\frac{M_c}{T_c} \qquad \text{and} \qquad P(R_c) < K_4\,\frac{{T_c}^4}{{M_c}^2}$$

with $K_3 = 4K_2\,/\,3K_1$ and $K_4 = {K_3}^{-3}(K_1 - K_2/K_3)$.

Schönberg-Chandrasekhar Limit – IV

From the earlier estimate of the minimum pressure at the centre of a star (P_c) and assuming that homology is applicable

$$P_c \propto \frac{G\,M^2}{8\,\pi R^4}$$

and by the Equation of State

$$P_c = \frac{k}{m_H \bar{\mu}}\,\rho_c\,T_c$$

it follows that

$$P \propto P_c \propto \frac{M}{R^3}\,T_c.$$

Combining with the proportionality at the top of the slide it can be seen that

$$T_c \propto \frac{M}{R} \qquad \text{and therefore} \qquad P \propto K_5\,\frac{{T_c}^4}{M^2}$$

Thus, the pressure at the surface of an isothermal core is independent of the core radius (R_c), but is inversely proportional to ${M_c}^2$ (from hydrostatic equilibrium) and M^2 (from homology).

Schönberg-Chandrasekhar Limit – V

Combining the two gives

$$\frac{M_c}{M} < \left(\frac{K_4}{K_5}\right)^{1/2}$$

which is the Schönberg-Chandrasekhar Limit (q_{SC}). It turns out that

$$q_{SC} \simeq 0.37 \left(\frac{\bar{\mu}_{env}}{\bar{\mu}_{core}}\right)^2$$

where $\bar{\mu}_{env}$ and $\bar{\mu}_{core}$ are the mean molecular weights for the envelope and core respectively. For an H-envelope $\bar{\mu}_{env} \simeq 0.6$ as already shown; for an He-core $\bar{\mu}_{core} \simeq 1.3$, giving $q_{SC} \simeq 0.1$.

In other words, under the assumption of virial equilibrium and homology, there is a maximum limit to the mass fraction of an isothermal core. Once $M_c/M \sim 0.1$, no stable configuration exists and the star must adopt a different equilibrium.

Schönberg-Chandrasekhar Limit – VI

On leaving the Main Sequence:

- If convective overshooting is neglected, stars having $M \lesssim 8.0\,M_\odot$ will leave the Main Sequence with an isothermal He-core satisfying $M_c(\mathrm{He}) < q_{\mathrm{SC}}$ and can maintain both hydrostatic and thermal equilibrium.
- Overshooting increases $M_c(\mathrm{He})$ at the end of central H-burning. With moderate overshooting $M_c(\mathrm{He}) < q_{\mathrm{SC}}$, and hydrostatic and thermal equilibrium are only maintained, if $M \lesssim 2.0\,M_\odot$.
- Massive stars will exceed q_{SC} on leaving the Main Sequence, and lower mass stars will do so after a period of H-shell burning during which $M_c(\mathrm{He})$ steadily increases.
- Once q_{SC} has been exceeded, thermal equilibrium is no longer possible.
- The He-core then contracts and establishes a temperature gradient which adds to the pressure gradient needed to balance gravity and keep the star in hydrostatic equilibrium.
- The He-core temperature gradient causes an outward heat flow such that (by the Virial Theorem) it keeps contracting and heating up.
- Contraction occurs on the thermal (Kelvin-Helmholtz) timescale in a quasi-static way, always maintaining a state very close to hydrostatic equilibrium.

Schönberg-Chandrasekhar Limit – VII

When low-mass ($M \lesssim 2\,M_\odot$) stars leave the Main Sequence:

- There is another way of maintaining both hydrostatic and thermal equilibrium.
- The He-core is relatively dense and cool; electron degeneracy can become important.
- Once a core becomes electron degenerate, $q_{\rm sc}$ no longer applies because degeneracy pressure, even in a relatively massive core, can support the pressure exerted on it by the envelope.
- Electron conduction within degenerate He-cores keeps them almost isothermal.

Core Contraction Effects

The following principle appears to be generally valid, and provides a way of interpreting the results of detailed numerical calculations:

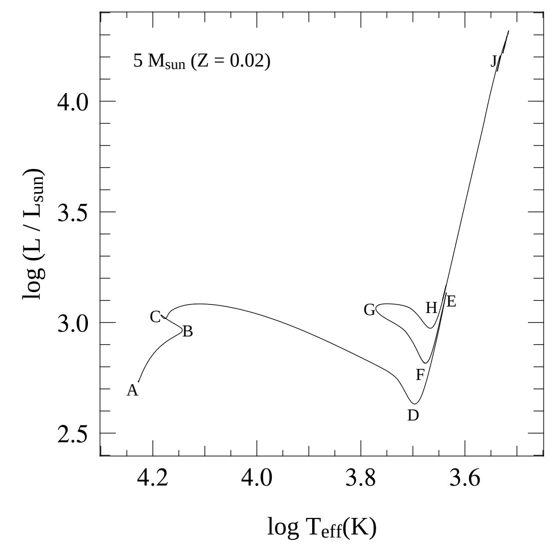
- Whenever a star has an active shell-burning source, the burning shell acts as a "mirror" between the core and envelope.

$$\text{core contraction} \Longrightarrow \text{envelope expansion}$$

$$\text{core expansion} \Longrightarrow \text{envelope contraction}$$

- The result may be understood by the following argument:
  - To maintain thermal equilibrium, the burning shell must remain at approximately constant temperature due to the thermostatic action of nuclear burning.
  - Contraction of the burning shell would also entail heating and so it must also remain at roughly constant radius.
  - If a core contracts, density and therefore pressure in the shell must decrease to maintain constant temperature and radius.
  - Pressure of the overlying envelope must also decrease.
- Envelope expansion therefore acompanies core contraction.

Intermediate and Massive Star H-Shell Burning – I



Intermediate and Massive Star H-Shell Burning – II

Qualitative differences are to be expected between low-mass ($M \lesssim 2\,M_\odot$) and intermediate-mass or massive ($M \gtrsim 2\,M_\odot$) stars. The two cases are discussed separately by reference to a detailed evolution calculation for $5\,M_\odot$ and $1\,M_\odot$ cases.

The previous slide shows the evolution track for the $5\,M_\odot$ case and the following slide shows some time-dependent interior details. Labels A, B, C, D, E, F, G, H and J in both diagrams are referenced to describe stages in the post-main sequence evolution.

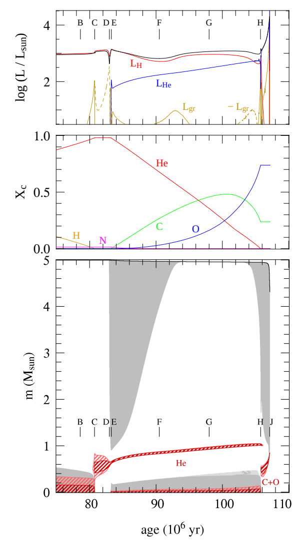
A) ZAMS

B) Start of the overall contraction phase near the end of the Main Sequence (when $X_c \simeq 0.03$).

C) H becomes exhausted in the centre and the convective core disappears; this is the point at which there is a rapid transition from H-burning in the centre to H-burning in a shell.

The H-exhausted core initially has a mass of about $0.4\,M_\odot < q_{\rm SC}$ and so the star initially maintains thermal equilibrium and the first portion of the H-shell burning phase (C-D) is relatively slow, lasting about 2×10^6 yr.

Intermediate and Massive Star H-Shell Burning – III



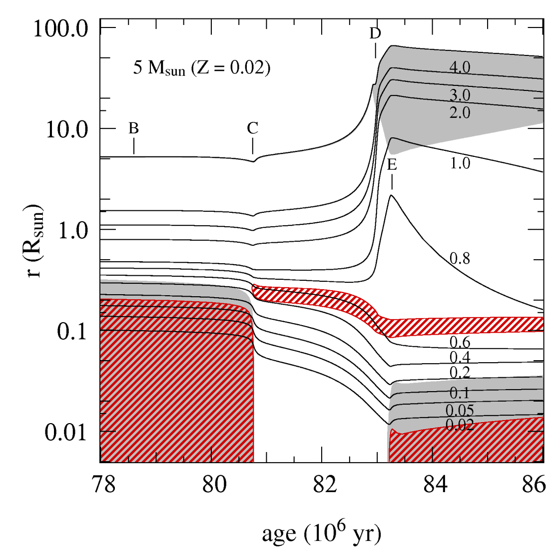
Intermediate and Massive Star H-Shell Burning – IV

- Temperature and density gradients between core and envelope remain shallow so that burning initially occupies a relatively large region in mass; this phase is therefore called **thick shell burning**.
- The He-core gradually grows in mass and eventually exceeds $q_{\rm SC}$ whereupon core-contraction speeds up.
- At the same time, the envelope expands as previously discussed.
- After C (in preceeding slide diagrams), layers below the burning shell contract while those above expand; this occurs at an accelerating rate towards the end of phase C-D.
- As a result, temperature and density gradients between the core and envelope increase, and the burning shell becomes thinner and thinner in mass.
- The latter stage of H-shell burning is therefore referred to as **thin shell burning**.
- Most of the time between C and D is spent in the thick shell burning phase at relatively small radii and $\log T_{\rm eff} > 4.05$.
- The phase of expansion from $\log T_{\rm eff} \simeq 4.05$ to D at $\log T_{\rm eff} \simeq 3.7$ occurs on the Kelvin-Helmholtz timescale and takes only a few times 10^5 yrs.

Intermediate and Massive Star H-Shell Burning – V

- A substantial fraction of energy generated by shell burning is absorbed by the expanding envelope, resulting in a decrease of the surface luminosity between C and D.
- Rapid evolution on a thermal timescale across the HRD from end of the Main Sequence to $T_{\rm eff} \simeq 5000\,{\rm K}$ is characteristic of intermediate-mass stars.
- The probability of detecting stars during this short-lived phase is very small, resulting in a gap in the distribution of stars in the HRD known at the **Hertzsprung Gap**.
- Envelope temperature decreases and opacity increases as D is approached, as a result ∇ increases.
- The envelope becomes increasingly unstable to convection, starting from the surface until at D a large fraction of the envelope mass is convective.
- During phase D-E the star is a red giant with a deep convective envelope and is located close to the Hayashi line in the HRD, while its envelope continues to expand in response to core contraction.
- Luminosity increases as the effective temperature remains at an approximately constant value corresponding to the Hayashi line.
- Expansion of the star between D and E still occurs on the thermal timescale, so the H-shell buring phase on the Red Giant Branch (RGB) is very short-lived.

Intermediate and Massive Star H-Shell Burning – VI



Intermediate and Massive Star H-Shell Burning – VII

- At its deepest extent at E, the base of the convective envelope is located at $M(r) = 0.9\,M_\odot$ which is below the maximum extent of the former convective core during central H-burning ($\sim 1.3 M_\odot$) at the ZAMS.
- Hence material formerly inside the convective core and processed by H-burning (CNO-Cycle) is mixed throughout the envelope and appears on the surface.
- The process is known as **dredge-up** and occurs about halfway between D and E (see previous slide).
- Helium cores of intermediate-mass stars remain non-degenerate during the entire H-shell burning phase C-E.
- Helium cores are developed with masses larger than $0.3 M_\odot$.
- A $5 M_\odot$ star at E has $M_c(\mathrm{He}) = 0.6\mathrm{M}_\odot$ when $T_c = 10^8\,\mathrm{K}$ is reached and He is ignited.
- He ignition halts further core contraction and envelope expansion and therefore corresponds to a local maximum in L and R.

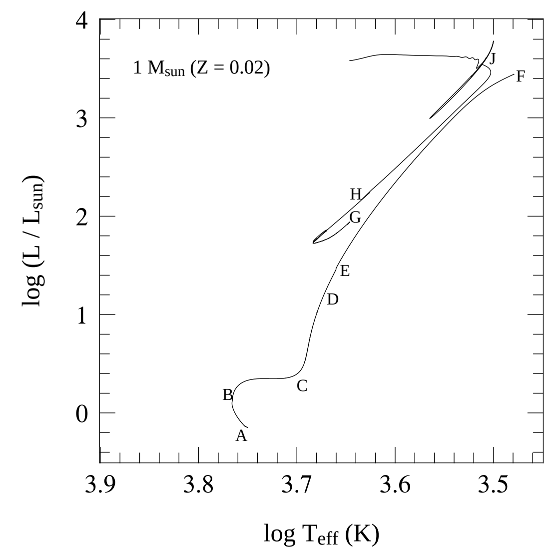
Low-Mass Star H-Shell Burning – I

Low-mass ($M \lesssim 2\,M_\odot$) have small or no convective cores during central H-buring; on leaving the Main Sequence these cores are close to becoming degenerate.

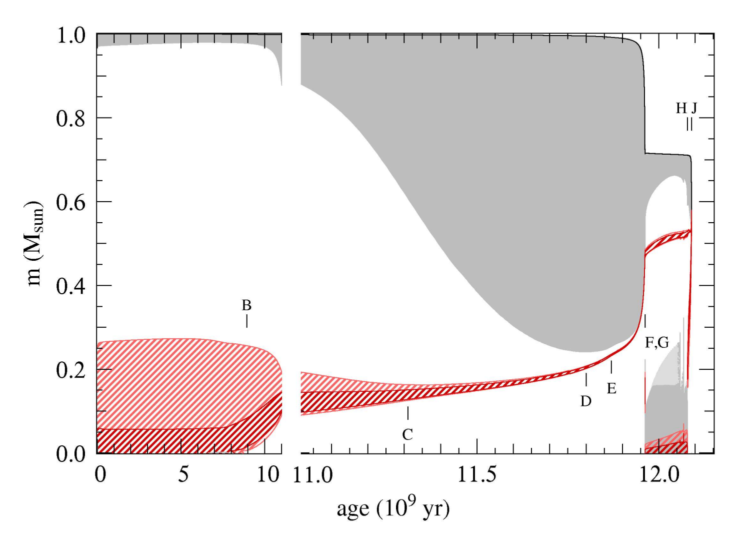
For stars having ($M \lesssim 1.1\,M_\odot$):

- The transition from central to shell H-burning is gradual.
- Initially $M_\mathrm{c}/M < 0.1$ so the star can remain in thermal equilibrium with an isothermal helium core.
- When the core has grown to $\simeq 0.1\,M$, its density is large enough for electron degeneracy to dominate the pressure and q_SC is irrelevant.
- Hydrostatic and thermal equilibrium is maintained throughout H-shell burning and there is no Hertzsprung Gap in the HRD.

Low-Mass Star H-Shell Burning – II



Low-Mass Star H-Shell Burning – III



Low-Mass Star H-Shell Burning – IV

The previous two slides show the evolution track and time-dependent interior details for the $1\,M_\odot$ case. Labels A, B, C, D, E, F, G, H and J in both diagrams are referenced to describe stages in the post-main sequence evolution.

- Hydrogen is practically exhausted in the centre ($X_c = 10^{-3}$) at B after 9 Gyr.
- Energy generation gradually moves out to a thick shell surrounding the isothermal helium core.
- Between B and C the core slowly grows in mass and contracts while the envelope expands.
- In response the H-burning shell becomes thinner in mass.
- By point C, the He core has become degenerate.
- At the same time the envelope has cooled and become largely convective, and the star is at the base of the RGB close to the Hayashi line.
- The star remains in thermal equilibrium throughout this B-C phase which lasts ~ 2 Gyrs for this $1\,M_\odot$ model; this corresponds to well-populated subgiant branches in old star cluster colour-magnitude diagrams.

Low-Mass Star H-Shell Burning – V

For stars having $1.1 \lesssim M \lesssim 1.5\,M_\odot$:

- Evolution and changes to internal structure are very similar to the $M \lesssim 1.1\,M_\odot$ case.
- The difference is that a small convective core is developed during core H-burning.
- As a result a "hook" occurs in the evolution track at central H exhaustion because the transition to H-shell burning is discontinuous.
- Subsequent evolution during H-shell burning is similar, the core remaining in thermal equilbrium until it becomes degenerate on the RGB after a slow evolution along the subgiant branch.

For stars having $1.5 \lesssim M \lesssim 2.0\,M_\odot$:

- Before He-cores become degenerate, the $q_{\rm sc}$ is reached.
- As a result, there is an Hertzsprung Gap in their evolution tracks which is small because the Main Sequence $T_{\rm eff}$ is close to the Hayashi Line.
- After a period of slow thick-shell burning on the subgiant branch, there is a period of rapid thermal-timescale expansion until the RGB is reached.

Low-Mass Star Red Giant Branch Evolution – I

A feature common to the post-main sequence evolution of all $M \lesssim 2\,M_\odot$ is that their He-cores become degenerate before T_c is high enough for He-ignition and they settle into thermal equilibrium on the RGB.

RGB Evolution in $M \lesssim 2\,M_\odot$ Stars:

- Evolution is very similar and almost independent of M.
- The reason is that by the time the He-core becomes degenerate, there is very high density contrast between the core and the envelope.
- Because the envelope is so extended, it exerts very little pressure on the compact core although there is a very large pressure gradient between the core and envelope.
- Pressure at the bottom of the envelope is very small compared with pressure at the edge of the core and in the H-burning shell separating the core and envelope.
- Stellar structure therefore depends almost entirely on properties of the He-core.
- Since the core is degenerate, its structure is independent of its thermal properties (temperature) and only depends on its mass.

Low-Mass Star Red Giant Branch Evolution – II

- Therefore the structure of a $M \lesssim 2\,M_\odot$ red giant is essentially a function of its M_c.
- As a result there is a very tight relation between the He-core mass and L, which is entirely due to H-shell burning.
- This $M_c - L$ relation is very steep for small core masses ($M_c \lesssim 0.5\,M_\odot$) and can be approximately described by a power law

$$L \simeq 2.3 \times 10^5 \left(\frac{M_c}{M_\odot}\right)^6 L_\odot.$$

- That is, the L of a $M \lesssim 2\,M_\odot$ red giant is independent of its total mass.
- Therefore the evolution of all $M \lesssim 2\,M_\odot$ stars converges, with ρ_c and T_c following the same evolution track.
- In the HRD the $M \lesssim 2\,M_\odot$ red giant is located near the Hayashi line appropriate for its M, meaning higher mass red giants have slightly higher $T_{\rm eff}$ at the same luminosity.
- As a consequence there is a $M_c - R$ relation, but it is not as tight as the $M_c - L$ and depends slightly on M.

Low-Mass Star Red Giant Branch Evolution – III

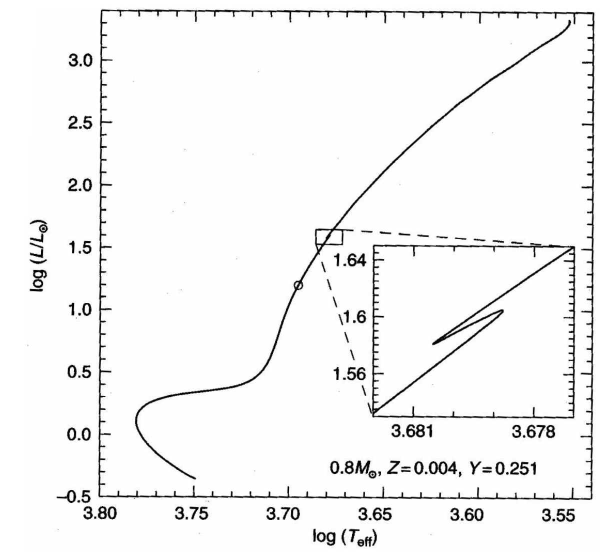
- As more He is added to the degenerate He-core by the H-burning shell, the core slowly contracts while R and L increase.
- A higher L means the H-shell must burn at a higher rate leading to a higher rate of increase in $M_{\rm c}$.
- Evolution up the RGB then goes faster as L increases.
- The density contrast between core and envelope increases and the mass within the H-burning shell decreases to $0.001\,M_\odot$ near the RGB tip.
- Since the H-burning shell mass decreases as L increases, the energy production rate per unit mass (ϵ) must increase dramatically, which implies a T increase within the burning shell.
- With an increasing T in the H-burning shell, T in the He-core must also increase.
- When the RGB tip is reached at F when $L \simeq 2000\,L_\odot$ and $M_{\rm c} \simeq 0.45\,M_\odot$, T in the degenerate He-core has reached $\simeq 10^8$ K and He is ignited.
- He-ignition is unstable due to degeneracy and leads to a thermonuclear runaway known as the **helium flash**.

Low-Mass Star Red Giant Branch Evolution – IV

First Dredge-Up and the Luminosity Bump:

- When the convective envelope reaches its deepest extent at D, it has penetrated into layers processed by H-burning during the Main Sequence and have been partly processed by the CN-Cycle.
- Up to D the surface He-abundance increases and the H-abundance decreases. More noticeably the C/N ratio decreases by a large factor; this is called the **First Dredge-Up** phase.
- Later at E, the H-burning shell has burnt its way out to the discontinuity left by the convective envelope at its deepest extent.
- The H-burning shell is then quite suddenly in an environment with a higher H-abundance (and lower mean molecular weight).
- As a result, the H-burning shell burns at a slightly lower rate leading temporarily to a slightly lower luminosity as shown in the next slide.
- The resulting loop (the star crosses this luminosity range three times) results in a larger number of stars in this luminosity range in a stellar population.

Low-Mass Star Red Giant Branch Evolution – V



Low-Mass Star Red Giant Branch Evolution – VI

Mass-loss is another process that becomes important in low-mass red giants,

- As L and R increase as a star evolves up the RGB, the envelope becomes more and more loosely bound and it is then relatively easy for the large photon flux to carry mass away from the stellar surface.
- The process driving mass-loss in red giants is not well understood.
- An empirical formula due to Reimers

$$\dot{M} = -4 \times 10^{-13}\,\boldsymbol{\eta}\,\frac{L}{L_\odot}\,\frac{R}{R_\odot}\,\frac{M_\odot}{M}\quad M_\odot/\mathrm{yr}$$

  is often used when calculating the effect of mass-loss in red giant evolution models. Here $\boldsymbol{\eta}$ is a parameter of order unity.
- To explain the morphology in the HRD of stars in the subsequent helium-burning phase, on the **Horizontal Branch**, a value of $\boldsymbol{\eta} \sim 0.25 - -0.50$ is often used to give the right amount of mass-loss.
- A $1\,M_\odot$ star looses $\sim 0.3\,M_\odot$ by the time it reaches the tip of the RGB.

Summary

Subjects discussed in the Seventh Lecture include:

- The fraction of stellar mass that may reside in a non-degenerate core under conditions of hydrostatic equilibrium.
- Core contraction implies envelope expansion and vice-versa.
- Evolution of a $5\,M_\odot$ star is followed from the end of the Main Sequence to the start of core helium burning.
- Evolution of a $1\,M_\odot$ star is also followed from the end of the Main Sequence to the start of core helium burning.
- The essential difference between the $5\,M_\odot$ and $1\,M_\odot$ cases is that the former ignites helium in a non-degenerate core.
- Helium burning in a $1\,M_\odot$ begins with the so called Helium Flash which results in a dramatic reduction in luminosity and stellar radius, as discussed in the next lecture.

Acknowledgement

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