4th Year Astrophysics: Galaxy Formation & Evolution Lecture Notes

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1 Preamble

Special thanks to Sadegh Kochfar who kindly provided access to his version of the lectures he gave. Although I perused many text books to write these notes, most of the material presented here (except the hierarchical galaxy formation lecture) is heavily drawn from "Galaxies in the Universe" by L.S. Sparke and J.S. Gallagher.

2 Basic definitions and properties of Galaxies

2.1 What is a galaxy?

What is a galaxy? When one views the zoo of pictures of galaxies available from e.g. the Hubble Heritage Image Gallery[1], this is presumably the first question that comes to mind. The answer is far from trivial, but a quite general definition could be: a gravitationally bound system composed of dark matter, stars, gas, and dust. Of course, some of these components are subdividible, with stars also comprising stellar remnants (black holes to white dwarfs) for instance, and gas is included in all its states (ionized, neutral and molecular). Supermassive black holes located at the centres of most, if not all, galaxies with bulges should not be forgotten as major players in galaxy evolution. Finally, dust is listed here as a separate component because of its key role in redistributing the electromagnetic radiation emitted by galaxies, radiation which constitutes most of the information on galaxies available to us.

2.2 Basic definitions

Since we know galaxies mostly through the light they emit, it makes sense to define their luminosity first. The *luminosity* L of a galaxy is the amount of energy it emits per second (unit W or erg s^{-1}). Their *flux* F, or apparent brightness is the amount of energy received per second per unit area on the observer's telescope (unit W m^{-2} or erg s^{-1} cm^{-2}). If the emission of light is isotropic, then $F \equiv L/(4\pi d^2)$, where d is the distance from the galaxy

[1] http://heritage.stsci.edu/gallery/galindex.html

to the observer. Related to fluxes are apparent magnitudes which are intrisically relative measures, i.e. obtained by comparing the brightness of an object with that of a reference star (or group of stars). The *apparent magnitude* of a galaxy m is then given by:

$$m - m_\star \equiv -2.5 \log_{10} \left(\frac{F}{F_\star} \right)$$

with $m_\star$ the magnitude of the reference star(s). Speeds of stars or galaxies are obtained by measuring the Doppler shift z in their spectral lines $z = \lambda_{obs}/\lambda_{em} - 1 = V_r/c$ where the last equality holds if the radial velocity V_r of the object is much smaller than the speed of light c. Speeds are generally *heliocentric*, i.e. given with respect to our Sun.

We rarely have access to the *bolometric* or total flux of galaxies. Instead, we measure their flux in a range of wavelengths, so we define a flux per unit wavelength F_λ (or per unit frequency F_ν) by setting $F_\lambda \Delta\lambda$ to be the energy received in $[\lambda, \lambda + \Delta\lambda]$ (and similarly for $F_\nu \Delta\nu$). Conservation of energy in matching intervals of wavelengths and frequencies implies that $F_\lambda = \nu^2/cF_\nu$. Units for F_λ are W m^{-2} Å^{-1} (10^{-26} W m^{-2} Hz^{-1} for F_ν). Finally, we have:

$$F = \int_0^\infty F_\nu d\nu = \int_0^\infty F_\lambda d\lambda.$$

Astronomers generally sample portions of the spectra of galaxies using *filter bandpasses* $\tau_{BP}(\lambda)$, which are defined by the fraction of light that they transmit at a given wavelength λ: $0 \leq \tau_{BP}(\lambda) \leq 1$. The flux of a galaxy in a given bandpass is then obtained through:

$$F_{BP} = \int_0^\infty \tau_{BP}(\lambda) F_\lambda(\lambda) d\lambda \approx F_\lambda(\lambda_{\text{eff}}) \Delta\lambda$$

where λ_{eff} is the *effective wavelength* of the filter, and $\Delta\lambda$ its width. Apparent magnitudes in this bandpass are:

$$m_{BP} - m_{\star,BP} = -2.5 \log_{10} \left(\frac{F_{BP}}{F_{\star,BP}} \right).$$

Historically, Vega was the reference star and by convention had zero magnitudes in all bandpasses, which is quite inconvenient because it means that the *zero point* is different in every bandpass. Now, a more commonly used system (e.g. the Sloan Digital Sky Survey) is the so called 'flux based' or AB system where

$$m_{BP} \equiv -2.5 \log_{10} \left(\frac{\langle F_{BP} \rangle}{\langle F_{V,0} \rangle} \right).$$

with

$$\langle F_{BP} \rangle = \frac{\int_0^\infty \tau_{BP}(\lambda) F(\lambda) d\lambda}{\int_0^\infty \tau_{BP} d\lambda}$$

and $\langle F_{V,0} \rangle = 3.63\, 10^{-9}$ erg s^{-1} cm^{-2} Å^{-1} so that when $\langle F_{BP} \rangle$ is measured in erg s^{-1} cm^{-2} Å^{-1}, $m_{BP} = -2.5 \log_{10} \langle F_{BP} \rangle - 21.1$, in **all** bandpasses. When astronomers talk about galaxy *colors*, such as $B - V$, they mean the difference between magnitudes in different bandpasses, i.e. $B - V = m_B - m_V$. They also refer to *absolute magnitude* M, which is the apparent magnitude an object would have if it was located 10 pc away from the observer, so that:

$$M = m - 5\log_{10}\left(\frac{d}{10\text{pc}}\right)$$

Finally, another key quantity for the study of galaxies is their *surface brightness* $I(\vec{x})$, defined as the amount of light emitted per squared arcsecond on the sky at point $\vec{x}$. Consider a small square patch of size D on a side, in a galaxy located at a distance d from the observer, so that it subtends an angle $\alpha = D/d$ and combine the luminosity of all stars there:

$$I(\vec{x}) \equiv \frac{F}{\alpha^2} = \frac{L/(4\pi d^2)}{D^2/d^2} = \frac{L}{4\pi D^2}$$

(units: mags arcsec^{-2} or L$_\odot$ pc^{-2}) which does not depend on the distance d to the object ... but see subsection 2.3 ... Contours of equal surface brightness are called *isophotes*.

2.3 Living in an expanding Universe

It must be pointed out that the expansion of the Universe was postulated for the first time by E. Hubble in 1929 on the basis of the measurement of the radial velocity of 22 galaxies. Hubble realized that they were all moving away from us at a radial speed proportional to their distance:

$$V_r = H_0 d$$

where $H_0 = 70 \pm 7$ km s^{-1} Mpc^{-1} is the current best estimate of the *Hubble constant*. Unless otherwise mentioned, a 0 subscript in these lectures refers to the present value of a function. A well established description of the expanding universe is given by the hot Big Bang model. This model has been studied in Lectures 1 to 5 by Dr. Ferreira, so we just recall here its key features, starting with the Friedmann-Robertson-Walker metric of an homogeneous and isotropic Universe:

$$ds^2 = c^2dt^2 - a(t)^2\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right) \quad (1)$$

where a stands for the *expansion factor* and k is a constant which gives the curvature of the Universe ($k = 0$ flat, $k = 1$ closed, $k = -1$ open). Note that $H_0 = [\dot{a}/a]_0$. Since Einstein's General Relativity states that spacetime tells matter how to move and in return matter tells spacetime how to curve, we can define a *critical density*

$$\rho_c(z) = \frac{3H^2(z)}{8\pi G}$$

where $z \equiv \lambda_{obs}/\lambda_{em} - 1 = a_0/a(t_{em}) - 1$ is the cosmological *redshift*, as the density for which the Universe is exactly flat. We can now compare all (four) energy densities

$\rho_i(z)$ to this critical value, and write the ratios as dimensionless functions of redshift, $\Omega_i(z) = \rho_i(z)/\rho_c(z)$. Current best estimates for these are $\Omega_m(0) = 0.27$, $\Omega_\Lambda(0) = 0.73$, $\Omega_k(0) = 0$, $\Omega_r(0) = 3 \times 10^{-5}$, where subscripts indicate contributions from matter, "dark energy", curvature and radiation respectively. Note that their sum is always equal to 1, and that only Ω_k can be negative. The second key feature of the Big Bang model which will be useful to us is the equation governing the expansion rate of the Universe:

$$H^2(z) = \left(\frac{\dot{a}}{a}\right)^2 = H_0^2 \left(\Omega_r(0)(1+z)^4 + \Omega_m(0)(1+z)^3 + \Omega_k(0)(1+z)^2 + \Omega_\Lambda(0)\right) \quad (2)$$

The fact that the Universe expands modifies the simple definitions given in subsection 2.2. In particular, the simple relation between flux and luminosity of a galaxy $F = L/(4\pi d^2)$ is not valid anymore, since the expansion saps the energy of the photons, and the area of the sphere over which these photons are spread expands. If r_{em} is the co-moving distance separating the observer and the galaxy at the time of emission t_{em} of the photons by the galaxy, then these photons would be spread over an area $4\pi a_{em}^2 r_{em}^2$, should we receive them instantaneously. However by the time we actually do receive them, they are spread over an area $4\pi a_0^2 r_{em}^2$ due to the expansion of the Universe. Moreover, the energy of these photons also has decayed by a factor $(1+z)$ (this is the cosmological red-shift defined above) and the time interval between two photons has increased by the same amount, since the total energy $L[t_{em}]\delta t_{em}$ emitted by the source between t_{em} and $t_{em}+\delta t_{em}$ must be conserved. In other words, we must have: $L[t_{obs}]\delta t_{obs} = L[t_{em}]\delta t_{em}$, i.e. $L[t_{em}]/(1+z)\delta t_{obs} = L[t_{em}]\delta t_{em}$, so $\delta t_{obs} = (1+z)\delta t_{em}$. These considerations lead us to rewrite the flux as (recalling that the luminosity is the amount of energy emitted *per unit time*):

$$F = \frac{L[t_{obs}]\delta t_{em}/\delta t_{obs}}{4\pi a_o^2 r_{em}^2} = \frac{L}{4\pi a_o^2 r_{em}^2 (1+z)^2}$$

where we have simplified the notation using $L[t_{em}] = L$ for the last equality. It is then natural to define the *luminosity distance* as $d_L = a_0 r_{em}(1+z)$. Similarly, at time t_{em}, a galaxy of size D subtends an angle α on the sky given by:

$$\alpha = \frac{D}{a_{em} r_{em}} = \frac{D(1+z)}{a_0 r_{em}}$$

so that we naturally define the *angular distance* as $d_A = a_0 r_{em}/(1+z)$. Note that, as a consequence of these changes, the surface brightness of a galaxy is now:

$$I(\vec{x}) = \frac{F}{\alpha^2} = \frac{L}{4\pi D^2} \frac{1}{(1+z)^4}$$

To calculate d_L and d_A, one can use equations 1 (for a photon $ds^2 = 0$) and 2 to estimate $a_0 r_{em}$. In the most general case, the result is not analytic, but for a critical *Einstein-de Sitter* (EdS) Universe with $\Omega_m = 1$ and all other contributions to the energy density nil, one gets:

$$a_0 r_{em} = \frac{2c}{H_0}\left(1 - \frac{1}{\sqrt{1+z}}\right)$$

3 The Basis of Hierarchical Galaxy Formation: growth and collapse of an isolated 'top hat' dark matter density perturbation

Let us consider a small perturbation in the otherwise homogeneous dark matter (collisionless fluid) density field of the Universe at some early time t_i (equivalent to high redshift z_i) after the decoupling of matter and radiation whose *peculiar velocity* $\vec{v_i} = \vec{0}$, i.e. the perturbation exactly follows the expansion of the Universe at t_i [2]. Its density contrast is $\delta_i \equiv (\rho(t_i) - \rho_m(t_i))/\rho_m(t_i) \ll 1$ where $\rho_m(t_i)$ is the density of the homogeneous Universe at time t_i (see subsection 2.3). From Dr Ferreira's lectures, the growth of this density fluctuation in the linear regime ($\delta(z) < 1$) is given by:

$$\delta(z) = \left\{\frac{3}{2}\Omega_m(0)(1+z_i) + 1 - \Omega_m(0) - \Omega_\Lambda(0)\right\} H_0^2 a_0^2\, D_g[z]\, \delta_i$$

with

$$\begin{aligned} D_g[z] &= \frac{D_d[z]}{a_0^2}\int_z^\infty \frac{(1+x)}{D_d^3[x]}\mathrm{dx} \\ D_d[z] &= H_0\left\{\Omega_m(1+z)^3 + (1-\Omega_m-\Omega_\Lambda)(1+z)^2 + \Omega_\Lambda\right\}^{\frac{1}{2}} \end{aligned}$$

the growing and decaying modes respectively, and we dropped the argument (0) for the Ω functions.

Although the following reasoning holds in the most general of cases, we will from now on consider the case of an EdS Universe (i.e $\Omega_m(z) = 1$, $\Omega_\Lambda(z) = 0$) since all integrals have analytic solutions in this specific case. The density contrast of our perturbation simplifies to:

$$\delta[z] = \frac{3}{5}\,\frac{1+z_i}{1+z}\,\delta_i \tag{3}$$

and we can define the *linearly extrapolated density contrast* $\delta_0 \equiv \delta[0] = 3/5(1+z_i)\delta_i$ as the density contrast that our perturbation would have today if it had never ceased to grow linearly.

Moving on to the non-linear regime, Birkhoff's theorem (GR equivalent of Newton's second theorem given in the Galactic Dynamics section) tells us that a spherically symmetric homogeneous a.k.a. *top hat* density perturbation evolves as an independent Universe with a slightly different density. We therefore use the equations of motion:

$$\begin{aligned} \frac{d^2a[t]}{dt^2} &= -\frac{4\pi G\rho_m(t_i)r_i^3}{3a^2[t]} \\ \frac{d^2r[t]}{dt^2} &= -\frac{4\pi G\rho_m(t_i)(1+\delta_i)r_i^3}{3r^2[t]} \end{aligned} \tag{4}$$

[2] This assumption does not reduce the generality of the calculation since a simple change in t_i always allows one to start from such initial conditions.

for the background Universe and the perturbation respectively, where we used $r_i = r(t_i) = a_i = a(t_i)$. Note that in considering only an EdS Universe we have effectively dropped a term $\Lambda/3 \times a[t]$ from the first equation of group 5 and a similar term $\Lambda/3 \times r[t]$ from the second equation, where Λ is the cosmological constant. Multiplying these equations by $\dot{a}$ and $\dot{r}$ respectively, and integrating them with respect to time, one gets:

$$\begin{aligned} \frac{\dot{a}[t]}{2} - \frac{4\pi G\rho_m(t_i)r_i^3}{3a[t]} &= E_b \\ \frac{\dot{r}[t]}{2} - \frac{4\pi G\rho_m(t_i)(1+\delta_i)r_i^3}{3r[t]} &= E_p \end{aligned} \tag{5}$$

Without loss of generality, one can set $E_b = 0$ [3] and derive $E_p = -4\pi G\rho_m(t_i)\delta_i r_i^2/3$ using the initial conditions $r(t_i) = a(t_i)$ and $\dot{r}(t_i) = \dot{a}(t_i)$. We can then rewrite the second equation of group 5 as:

$$\dot{r}[t] = \sqrt{\Omega_m(z_i)H^2(z_i)r_i^2\left[(1+\delta_i)\frac{r_i}{r[t]} - \delta_i\right]}$$

Note that functions/constants/variables without the subscript p always refer to the background Universe by opposition to the perturbation, although $H_p(z_i) = H(z_i) = H_0(1+z_i)^{3/2}$ with the last equality being valid in an EdS Universe only. In such a Universe, any overdensity is bound because the background density already is at the critical density, i.e. that which divides open and closed Universes. This means that the expansion of our initially slightly overdense perturbation will become slower and slower with respect to the homogeneous background Universe, until time t_{max} when it reaches its radius of maximum expansion r_{max}, also called *turn around* radius, where $\dot{r}[t_{max}] = 0$. Plugging these conditions in the previous equation we get $r_{max} = (1+\delta_i)r_i/\delta_i$. Further separating variables t and r, and integrating after setting $u = \sqrt{r/r_i}$ and $\Omega_m(z_i) = \Omega_m(0) = 1$, yields:

$$\int_{t_i}^{t_{max}} dt = \frac{1}{H_0(1+z_i)^{3/2}} \int_1^{\sqrt{r_{max}/r_i}} \frac{2u^2 du}{\sqrt{1+\delta_i - \delta_i u^2}}$$

i.e.

$$t_{max} = t_i + \frac{1}{H_0(1+z_i)^{3/2}} \left\{ \frac{r_{max}}{r_i} - 1 + \frac{r_{max}}{r_i}\sqrt{\frac{r_{max}}{r_i} - 1}\left[\frac{\pi}{2} - \arcsin\sqrt{\frac{r_{max}}{r_i}}\right] \right\}$$

Since $r_{max}/r_i \gg 1$ and $t_{max} \gg t_i$, replacing r_{max} by its expression as a function of δ_i and r_i and using the definition of δ_0, we can recast the time of turn around as:

$$t_{max} \simeq \frac{\pi}{2H_0}\left(\frac{3}{5\delta_0}\right)^{3/2}$$

[3] We can pick the zero point we want for the perturbation, because we can always write $E_p = E_b + E'_p$ and consider the constant E'_p instead of E_p

Now our perturbation will collapse to a point in a time $t_{coll} = 2t_{max} - t_i \simeq 2t_{max}$ given the symmetry of the problem. In the homogeneous Universe, this time of collapse will correspond to the redshift of collapse z_{coll} defined by (see Dr Ferreira's lectures):

$$t_{coll} \equiv \int_{z_{coll}}^{\infty} \frac{1}{(1+x)D_d[x]}\,dx = \frac{2}{3H_0}\frac{1}{(1+z_{coll})^{3/2}}$$

Equating the two expressions obtained for t_{coll}, we can define $\delta_{0,c}$ as the *linearly extrapolated critical density contrast* that a perturbation must have in order to collapse **exactly** at redshift z_{coll}:

$$\delta_{0,c}[1+z_{coll}] = \frac{3(12\pi)^{2/3}}{20}(1+z_{coll}) \tag{6}$$

In other words, if the linearly extrapolated density contrast δ_0 of a perturbation with density contrast $\delta_i \ll 1$ at redshift z_i is equal to $3(12\pi)^{2/3}/20 \approx 1.686$, then this density perturbation will collapse at $z = 0$. If δ_0 is twice this value, it will collapse at $z = 1$, if it is four times greater, collapse will occur at $z = 3$ and so on and so forth.

Note that for an exactly homogeneous perturbation as we have assumed, the density becomes infinite at the center at z_{coll}, since all the mass arrives there at the same time. What it means is that the model is not a very good description of the reality of dark matter halo collapse: in the real Universe, none of the density fluctuations are exactly homogeneous (at least on galactic dark matter halo scales) and 'sub-halos' will collapse first and merge together later to form a bigger halo in a process that is called *violent relaxation* and happens under the only constraint of the integrals of motion. To cut a long story short, in that case, dark matter particles end up populating the available phase sphace equiprobably and the end (stationary) state for the halo is the most probable one. In the case where only mass and total energy are conserved, the final density profile is proportional to r^{-2} and is called *isothermal sphere* profile, and one can show that the *virial radius* of the halo is $r_{vir} \approx 0.5\, r_{max}$. The final density contrast of the halo at z_{coll} is therefore $\delta_h \approx 18\pi^2 \simeq 178$. To a large extent this scenario and numbers are corroborated by numerical $N-$body simulations.

4 An Introduction to Galactic Dynamics

4.1 Newton-Poisson equations

Unsurprizingly, the starting point to study the dynamics of a galaxy is *Newton's law of gravitation.* Applied to a system of N stars with masses m_i, positions $\vec{x_i}$ and velocities $\vec{v_i}$ where the subscript $i = 1, 2, ..., N$, it allows us to calculate the gravitational force exerted by the other stars on star i:

$$\frac{d(m_i\vec{v_i})}{dt} = -\sum_{j}^{j\neq i} \frac{Gm_i m_j}{|\vec{x_i} - \vec{x_j}|^3}(\vec{x_i} - \vec{x_j}) \tag{7}$$

Since the mass m_i cancels out in equation 7, the acceleration $d\vec{v_i}/dt$ of star i is independent of the star mass (principle of equivalence between inertial and gravitational mass which is the basis of Einstein's GR). We can rewrite equation 7 dropping subscript i, as:

$$\frac{d\vec{v}}{dt} = -\vec{\nabla}\phi(\vec{x}) \;; \quad \text{where } \phi(\vec{x}) = -\sum_j \frac{Gm_j}{|\vec{x}-\vec{x_j}|} \text{ with } \vec{x} \neq \vec{x_j} \tag{8}$$

and $\phi(\vec{x})$ is called the *gravitational potential*. Note that equation 8 implies that, in defining the gravitational potential we have chosen arbitrarily a constant of integration such that $\phi(\vec{x}) \rightarrow 0$ when $|\vec{x}| \rightarrow \infty$. If the distribution of matter in the galaxy is continuous (e.g. we wish to account for the presence of gas as well as stars) then the potential becomes

$$\phi(\vec{x}) = -\int_V \frac{G\rho(\vec{x'})}{|\vec{x}-\vec{x'}|} d^3x' \tag{9}$$

with the integration being performed over all space except in $\vec{x}$. Equation 9 can be turned into a differential equation by applying the Laplacian operator $\Delta = \nabla^2$ to both sides:

$$\nabla^2\phi(\vec{x}) = -\int_V G\rho(\vec{x'})\nabla^2\left(\frac{1}{|\vec{x}-\vec{x'}|}\right) d^3x'$$

differentiating in 3-D space with respect to $\vec{x}$, we find that $\nabla^2(1/|\vec{x}-\vec{x'}|) = 0$ so that the integrand of the term on the right hand side (RHS) is zero outside a small sphere $S_\epsilon(\vec{x})$ of radius ϵ centered on $\vec{x}$. In the limit $\epsilon \rightarrow 0$, $\rho(\vec{x})$ is almost constant within this sphere, which allows us to write:

$$\nabla^2\phi(\vec{x}) \simeq -G\rho(\vec{x})\int_{S_\epsilon(\vec{x})} \nabla^2_{\vec{x'}}\left(\frac{1}{|\vec{x}-\vec{x'}|}\right) d^3x' \tag{10}$$

where we have also swapped the variable over which the Laplacian is applied (this is legit since the function $1/|\vec{x}-\vec{x'}|$ is symmetric in $\vec{x}$ and $\vec{x'}$). Now we can use the divergence theorem to convert the volume integral into a surface integral:

$$\int_{S_\epsilon(\vec{x})} \nabla^2_{\vec{x'}}\left(\frac{1}{|\vec{x}-\vec{x'}|}\right) d^3x' = \int_{\Sigma_\epsilon(\vec{x})} \vec{\nabla}_{\vec{x'}}\left(\frac{1}{|\vec{x}-\vec{x'}|}\right) d\vec{S'}$$

where the integrand of the term on the RHS is $(\vec{x}-\vec{x'})/|\vec{x}-\vec{x'}|^3$, i.e. a vector of length ϵ^{-2} pointing in towards $\vec{x}$ and the surface area $\Sigma_\epsilon(\vec{x}) = 4\pi\epsilon^2$, so that the integral is simply -4π! As a result, equation 10 now reads:

$$\nabla^2\phi(\vec{x}) = 4\pi G\rho(\vec{x}) \tag{11}$$

and is called the *Poisson equation*. It is fundamental for galactic dynamics because it allows us to choose a mathematically convenient potential to describe the density field of a galaxy, being careful to enforce $\rho(\vec{x}) \geq 0$.

4.2 Spherically symmetric systems

Newton proved two very useful theorems about the gravitational field of such systems which are very useful to understand elliptical galaxies (at least to zeroth order):

1. The gravitational force *inside* a spherical shell of uniform density is zero
2. *Outside* any spherically symmetric object, the gravitational force is the same as if all the mass of the object had been concentrated at its center.

See classical textbooks e.g. "Galactic Dynamics" by J. Binney and S. Tremaine for a crystal clear presentation of the proof. These theorems tell us that within any spherically symmetric galaxy of density $\rho(r)$, the gravitational force towards the center at position $\vec{r}$ is simply the sum of the inward forces exerted by all the matter contained within radius r. In particular, the acceleration $V_c^2(r)/r$ of a star moving with *circular velocity* $V_c(r)$ on an orbit of radius r about the center must be provided by the inward gravitational force $-F_r(r)$. Writing $M_{<R}$ the mass enclosed within radius r, we therefore have

$$\frac{V_c^2(r)}{r} = -F_r(r) = \frac{GM_{<R}}{r^2}$$

Whenever we can find gas or stars in near circular orbits within a galaxy, this equation is by far the simplest and most reliable method to estimate the mass within the orbital radius! The potential $\phi(r)$ of spherically symmetric systems can be conveniently decomposed into two parts, one where radii $r' < r$ and the other where $r' > r$ as:

$$\phi(r) = -4\pi G\left[\frac{1}{r}\int_0^r \rho(r')r'^2 dr' + \int_r^\infty \rho(r')r' dr'\right]$$

which is *not* equal to $GM_{<R}/r$ unless all the mass of the system is located within r. However equation 9 implies that at a great distance from any system with finite mass M, $\phi(\vec{r}) \to -GM/r$.

When studying the orbit of a single star moving through a galaxy which contains billions of them (not to mention gas and dark matter), we can neglect the effect that this star changes the potential of the galaxy by attracting all the other stars. In other words, unless the galaxy is obviously collapsing or colliding with another we can consider its mass distribution as 'static' or equivalently its potential $\phi(\vec{r})$ as independent of time. Then, as the stars moves with velocity $\vec{v}$, $\phi(\vec{r})$ at its location $\vec{r}$ changes according to:

$$\frac{d\phi(\vec{r})}{dt} = \vec{v}\vec{\nabla}\phi(\vec{r})$$

Taking the scalar product of $\vec{v}$ with equation 8 we have:

$$\vec{v}\frac{d(m\vec{v})}{dt} + m\vec{v}\vec{\nabla}\phi(\vec{r}) = \frac{d}{dt}\left(\frac{1}{2}mv^2 + m\phi(\vec{r})\right) = 0$$

thus the total energy $E_{tot} = E_K + E_P = 1/2mv^2 + m\phi$ of the star is conserved along its orbit. Since $E_K \geq 0$ and that when we are far away from a galaxy $E_P \to 0$ (otherwise it stays negative), a star at position $\vec{r}$ can only escape if $E_{tot} > 0$, i.e. if it is moving faster than the local *escape velocity* $V_e \equiv \sqrt{-2\phi(\vec{r})}$. In a similar vein, the angular momentum of the star along its orbit $\vec{L} = m\vec{r} \times \vec{v}$ changes according to:

$$\frac{d\vec{L}}{dt} = \vec{r} \times \frac{d(m\vec{v})}{dt} = -m\vec{r} \times \vec{\nabla}\phi(\vec{r})$$

If the galaxy is spherically symmetric about $\vec{r} = \vec{0}$, the force $-\vec{\nabla}\phi$ is radial and therefore $\vec{r} \times \vec{\nabla}\phi = 0$, which means that $\vec{L}$ is conserved along the orbit.

4.3 Axisymmetric systems

If we are prepared to overlook non-axisymmetric features of spiral galaxies (such as bars and spiral arms), the smooth gravitational potential in which stars are moving along nearly (but not quite) circular orbits, almost in the same plane, is independent of the azimuthal angle θ of a cylindrical polar coordinate system centered on the galaxy, with z axis perpendicular to the galactic plane. Hence, $\partial\phi/\partial\theta = 0$ and there is no net force in the azimuthal direction, i.e. a star conserves its angular momentum about the z axis. That is the main difference with spherically symmetric galaxies where the **total** angular momentum of a star is conserved as it moves on its orbit. On writing L_z for the z angular momentum per unit mass for a star moving in an axisymmetric system we have:

$$\frac{d(R\dot{\theta})}{dt} = 0 \text{ so } L_z \equiv R^2\dot{\theta} = \text{constant}$$

Since the potential $\phi = \phi(R, z)$ is assumed to be independent of time, we can write the equation of motion of the star in the radial direction as:

$$\ddot{R} = R\dot{\theta}^2 - \frac{\partial\phi}{\partial R} = -\frac{\partial\phi_{eff}}{\partial R} \text{ where } \phi_{eff} \equiv \phi + \frac{L_z^2}{2R^2}$$

is the *effective potential* which behaves like a potential energy for the star's motion in R and z. Upon multiplying this equation of motion by $\dot{R}$ and integrating it with respect to time, we get, for a star moving in the midplane $z = 0$:

$$\frac{1}{2}\dot{R}^2 + \phi_{eff}(R, z = 0, L_z) = \text{constant}$$

Since $\dot{R}^2 \geq 0$ the L_Z^2 term in ϕ_{eff} acts like an "angular momentum barrier" to prevent a star with $L_z \neq 0$ from coming closer to $R = 0$ than some perigalactic radius where $\dot{R} = 0$. Unless it has a velocity which is large enough to escape from the galaxy a star must remain within some apogalactic outer limit. In other words its orbit is bounded.

In the vertical direction, the equation of motion of a star is given by:

$$\ddot{z} = -\frac{\partial\phi(R, z)}{\partial z} = -\frac{\partial\phi_{eff}}{\partial z}$$

If the 'top' and 'bottom' part of the disk are mirror images of each other, $\phi(R, z) = \phi(R, -z)$ and the force is zero in the midplane $z = 0$. Setting R_g to be the average value of R for the star's orbit, and expanding $\phi(R,z)$ in Taylor series around $(R_g, 0)$, keeping the leading term (i.e. errors will only be as large as z^2/R^2 or $(R-R_g)^2/R^2$) alone, one gets, for these nearly circular orbits:

$$\ddot{z} \approx -z \left[\frac{\partial^2 \phi(R_g, z)}{\partial z^2}\right]_{z=0} \equiv -\nu^2[R_g]z$$

This means that the motion of the star in the z direction is almost independent of that in R and θ. This equation is also that of an harmonic oscillator with angular frequency ν, i.e. the solution is $z = Z\cos(\nu t + \beta)$ where Z and β are constants. Note that in a flattened galaxy, $\nu(R)$ is larger than the angular speed $\Omega(R)$ on a circular orbit.

A star with angular momentum L_z can follow exactly a circular orbit with $\dot{R} = 0$, only at R_g where ϕ_{eff} is stationary with respect to R. There, the equation of motion of a star in the radial direction tells us that:

$$\frac{\partial \phi(R_g, z=0)}{\partial R} = \frac{L_z^2}{R_g^3} = R_g \Omega^2(R_g)$$

If ϕ_{eff} has a minimum at R_g then the circular path is the orbit with least energy for the given angular momentum L_z. It means that the orbit of the star is stable and that any star with the same L_z must oscillate around it. As the star moves in and out radially, its azimuthal motion must alternatively speed up and slow down to keep L_z constant. We can show that it approximately follows an elliptical *epicycle* around its *guiding center* which moves with angular speed $\Omega(R_g)$. To derive the epicyclic equation, we set $R = R_g + x$ in the equation of radial motion of the star with $x \ll R$ and we neglect second order terms when Taylor expanding:

$$\ddot{x} \approx -x \left[\frac{\partial^2 \phi_{eff}(R, z=0)}{\partial R^2}\right]_{R=R_g} \equiv -\kappa^2[R_g]x$$

so the solution is $x = X\cos(\kappa t + \psi)$ where X and ψ are constants. When $\kappa^2 > 0$ this equation describes an harmonic motion with *epicyclic frequency* κ. If $\kappa^2 < 0$, the circular orbit is unstable and the star moves away from it at an exponentially increasing rate. From the definition of ϕ_{eff}, and recalling that $R\Omega^2(R) = \partial\phi(R, z=0)/\partial R$ on a circular orbit, we obtain:

$$\kappa^2(R) = \frac{d(R\Omega^2(R))}{dR} + \frac{3L_z^2}{R^4} = \frac{1}{R^3}\frac{d(R^2\Omega(R)^2)}{dR} \equiv -4B\Omega$$

where B is *Oort's constant* named after Dutch astronomer Jan Oort who was the first to measure it in the solar neighborhood. Locally $B < 0$ so $\kappa^2 > 0$ and near circular orbits like that of our sun are fortunately stable!

As previously mentioned, during its epicyclic motion, the star's azimuthal speed $\dot{\theta}$ must vary so that L_z remains constant: $\dot{\theta} = L_z/R^2 = \Omega(R_g)R_g^2/(R_g+x)^2$, so $\dot{\theta} \approx \Omega(R_g)(1 - 2x/R_g)$ to leading order and substituting for x and integrating with respect to time we get:

$$\theta(t) = \theta_0 + \Omega(R_g)t - \frac{2\Omega(R_g)}{\kappa R_g}\chi\sin(\kappa t + \psi)$$

where θ_0 is an arbitrary constant. Here, the two first terms on the RHS give the guiding center's motion and the third term the harmonic motion of the star with the same frequency κ than the x oscillations in radius but 90 degrees out of phase and larger by a factor $2\Omega(R_g)/R_g$. Note that the epicyclic motion is retrograde, i.e. takes place in the opposite sense to the motion of the guiding center.

5 Main Galaxy Types: Spirals, S0s and Ellipticals

This section briefly describes the physics behind the fundamental scaling laws observed for galaxies of different types.

5.1 The Tully-Fisher Relation

We start by deriving the scaling observed between the maximum rotation velocity of the disks of spiral galaxies, V_{max}, and their luminosity L. Observations of rotation curves $V_{rot}(r)$ of spiral galaxies reveal that they rise sharply from $r = 0$, reach a maximum value V_{max} and remain roughly flat in their outer parts. Assuming that a star at the outer edge of the spiral is on a stable circular orbit around the centre of the galaxy, we can say that the centrifugal force acting on that star is equal to the gravitational pull exerted on it by the mass distribution enclosed within the sphere bounded by the orbit:

$$\frac{mV_{max}^2}{R} = \frac{GmM_{<R}}{R^2}, \tag{12}$$

where m is the mass of the star, R the radius of the galaxy, $M_{<R}$ the mass inside the sphere of radius R and G the gravitational constant. Rearranging Eq.12 we get an expression for the mass $M_{<R}$:

$$M_{<R} = \frac{V_{max}^2 R}{G}. \tag{13}$$

Please note that the mass $M_{<R}$ is not only the luminous baryonic mass in stars and gas, but also the dark matter enclosed within the sphere of radius R.

Let us now assume that spiral galaxies of one type (e.g. Sa, b, c) all have the same mass-to-light ratio $M/L \equiv 1/c_{ML}$. This of course is a simplification, reflecting our prejudice that spiral galaxies of the same type have the same intrinsic structure and star formation histories, and that small spirals are just scaled down versions of larger ones. Replacing the mass $M_{<R}$ in Eq. 13 by the luminosity L using the mass-to-light ratio leads to

$$L = \frac{c_{ML} V_{max}^2 R}{G}. \tag{14}$$

Finally, if spiral galaxies have a constant surface brightness $L/R^2 \equiv c_{SB}$ (so called Freeman's law), we can take the square of the right and left hand side terms of Eq. 14 and rewrite it as:

$$L = \frac{c_{ML}^2}{c_{SB}} \frac{V_{max}^4}{G^2} = CV_{max}^4. \tag{15}$$

where we have combined all constants into one by setting $C = c_{ML}^2/(c_{SB}G^2)$. This is called the Tully-Fisher relation and can be recast with magnitudes instead of luminosities to yield:

$$M = -2.5\log(L) = -10\log(V_{max}) + const, \tag{16}$$

which is close to the actual observed relation.

5.2 The Faber-Jackson Relation

We now proceed to derive a similar relation to the Tully-Fisher for spirals but which applies to elliptical galaxies. As the fundamental difference between the dynamics of disks and spheroids is that these latter are not supported by rotation, we replace our centrifugal force arguments by the assumption that spheroids are virialized to derive an expression for the scaling of their stellar velocity dispersion σ with their luminosity L:

$$\frac{1}{2}\left\langle \frac{d^2 I}{dt^2} \right\rangle - 2\left\langle K \right\rangle = \left\langle U \right\rangle. \tag{17}$$

where I is the moment of inertia, $\langle K\rangle$ the time-averaged kinetic energy and $\langle U\rangle$ the time-averaged potential energy of the stars. Elliptical galaxies appear to be dynamically relaxed systems which are close to equilibrium. If this is indeed the case $\langle d^2I/dt^2\rangle = 0$ and the usual form of the virial theorem emerges:

$$-2\left\langle K \right\rangle = \left\langle U \right\rangle. \tag{18}$$

For a large number of stars, N, the time averaging in the virial theorem 18 can be dropped because the system will look the same in a statistical sense. In other words, at each instant in time, the number of stars with kinetic energy K_i and potential energy U_i will remain the same on average. Rewriting Eq. 18 in terms of the sum of the kinetic energy of individual stars we get:

$$-2\sum_i \frac{1}{2} m_i v_i^2 = U. \tag{19}$$

For simplicity, let us consider the case of a perfectly spherically symmetric elliptical galaxy (type E0) composed of stars which all have the same mass m. The assumption about spherical symmetry allows us to handle the right hand term of Eq. 19, i.e. the potential energy, more easily, and the assumption of identical mass stars helps us simplify both left hand and right hand terms in Eq. 19. Using these simplifications and dividing Eq. 19 by N yields:

$$-\frac{m}{N}\sum_i v_i^2 = \frac{U}{N} \tag{20}$$

By definition of the velocity dispersion,

$$\frac{1}{N}\sum_i v_i^2 = \left\langle v^2 \right\rangle = \sigma^2 \tag{21}$$

and combining Eq. 21 and Eq. 20, one obtains the following expression [4]:

$$-m\sigma^2 = \frac{U}{N}. \tag{22}$$

We now proceed to derive an expression for the potential energy using an isotropic mass distribution. Consider the amount of mass dm locked in a spherical shell of stars of thickness dr located at a distant r from the centre of the galaxy

$$dm = 4\pi r^2 \rho(r) dr. \tag{23}$$

The potential energy of that shell reads:

$$dU = -G\frac{M_{<r}}{r} 4\pi r^2 \rho(r) dr. \tag{24}$$

To get the total potential energy one then simply needs to integrate over all shells:

$$U(R) - U(0) = U(R) = -4\pi G \int_0^R r M_{<r} \rho(r) dr. \tag{25}$$

It is worth noting that $\rho(r)$ is the radial density profile of the stars in the galaxy, while $M_{<R}$ is the mass of stars plus dark matter inside a given radius r, as a star will always feel the combined gravitational force of dark matter and other stars. However, since observations of elliptical galaxies reveal that in their inner parts they are dominated by baryonic matter we will assume in the following that $M_{<R}$ is the mass of stars and that the contribution of dark matter is negligible. For a given density profile $\rho(r)$ the mass inside a radius r can be readily calculated as:

$$M_{<r} = 4\pi \int_0^r r'^2 \rho(r') dr'. \tag{26}$$

A particularly convenient choice is to use a singular isothermal sphere density profile:

$$\rho(r) = \rho_0 \left(\frac{r_0}{r}\right)^2, \tag{27}$$

with ρ_0 and r_0 being constants characterising the profile. Even though such a density profile diverges at the center, Eq. 27 and Eq. 26 can be combined to calculate the total mass of the galaxy inside radius r:

$$M_{<r} = 4\pi \int_0^r \frac{r'^2}{r'^2} \rho_0 r_0^2 dr' = 4\pi \rho_0 r_0^2 r. \tag{28}$$

Inserting Eq. 27 and Eq. 28 into Eq. 25 the potential energy can be written as:

$$U = -4\pi G \int_0^R 4\pi \rho_0^2 r_0^4 \frac{r^2}{r^2} dr = -16\pi^2 G \rho_0^2 r_0^4 R. \tag{29}$$

[4] Note that the brackets in Eq. 21 do not mean that the velocity has to be averaged over time, but instead has to be averaged over all the stars

Using Eq. 28 for the total mass of the galaxy M in Eq. 29 one obtains the following expression for the total potential energy of an elliptical galaxy whose density profile is that of an isothermal sphere:

$$U = -\frac{GM^2}{R}. \tag{30}$$

Now we can insert Eq. 30 in Eq. 22 to derive an explicit form of the virial theorem in that case:

$$-m\sigma^2 = -\frac{GM^2}{RN}. \tag{31}$$

Further replacing Nm by the total mass M yields:

$$\sigma^2 = \frac{GM}{R}. \tag{32}$$

Assuming, as in the derivation of the Tully-Fisher Relation that the mass-to-light ratio of the elliptical galaxies is constant, i.e. $M/L \equiv 1/c_{ML}$, one gets:

$$\sigma^2 = \frac{GL}{c_{ML}R}. \tag{33}$$

Further assuming that their surface brightness is constant, i.e. that $L/R^2 \equiv c_{SB}$, and taking the square of both right hand and left hand terms in the previous equation leads to the so called Faber-Jackson relation:

$$\sigma^4 \propto L. \tag{34}$$

5.3 The Fundamental Plane of Elliptical Galaxies

Elliptical galaxies also live on a plane in the three dimensional space defined by the values of their surface brightness, radius and velocity dispersion. Once again, to derive the equation of this plane, the starting point is the virial theorem given in Eq. 32:

$$\sigma^2 \propto \frac{GM}{2R}. \tag{35}$$

Multiplying Eq. 35 by (LR/LR):

$$\sigma^2 \propto \left(\frac{M}{L}\right) R \left(\frac{L/2}{R^2}\right). \tag{36}$$

Moreover, if we assume that the radius of the galaxy is proportional to the effective radius R_0, defined as the radius within which half of the galaxy mass lies, one gets:

$$\sigma^2 \propto \left(\frac{M}{L}\right) R_0 I_0. \tag{37}$$

Here $I_0 \propto L/2R_0^2$ is the surface brightness at the effective radius. Taking the logarithm of Eq. 37 and rearranging its terms leads to the final relation:

$$\log R_0 = a \log \sigma + b \log I_0 + c, \tag{38}$$

which is called the fundamental plane, and where a, b and c are constants defining the position of the plane in the $R_0 - \sigma - I_0-$manifold. The fundamental plane thus appears to be a consequence of elliptical galaxies being virialized objects. Note that the best fit parameters to the observed fundamental plane slightly vary with wavelength.

6 Groups, Clusters and Gravitational Lensing

Galaxies are not randomly distributed on the sky, but tend to cluster together forming groups (containing a few to a few tens of galaxies) and clusters (containing a hundred to a few hundred of galaxies). Within these larger structures, galaxies spiral down to the center through dynamical friction and merge together to form the largest galaxies known in the Universe.

6.1 Collisionless Dynamics: Dynamical Friction

Consider a galaxy with mass M and velocity $\mathbf{v}_M$ moving through a medium of density ρ. One can show that the drag force exerted on this galaxy (the gravitational pull of the medium is stronger in the wake of the galaxy because of its movement: this effect is called dynamical friction as it only exists if the galaxy is in motion and it opposes this motion) is given by:

$$\mathbf{F}_{df} = -\frac{4\pi \ln \Lambda G^2 \rho M^2}{v_M^3} \left[\mathrm{erf}(X) - \frac{2X}{\sqrt{\pi}} e^{-X^2} \right] \mathbf{v}_M, \tag{39}$$

where $X \equiv v_M/\sqrt{2}\sigma$ and the error function is defined as

$$\mathrm{erf}(z) \equiv \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt. \tag{40}$$

$\ln \Lambda$ is called the Coulomb logarithm, and for the case of a galaxy *sat*, falling in cluster or a group *host* takes a value

$$\ln \Lambda \approx \ln(1 + M_{DM,host}/M_{DM,sat}) \tag{41}$$

where the DM subscript refers to the dominant dark matter collisionless fluid component of each system (host cluster and satellite galaxy respectively). Assuming these components are distributed with the density profile of an isothermal sphere

$$\rho(r) = \frac{v_c^2}{4\pi G r^2} \tag{42}$$

yields a velocity dispersion of $\sigma = v_c/\sqrt{2}$ and the drag force becomes:

$$\begin{aligned} F_{df} &= -\frac{4\pi \ln \Lambda G^2 M^2 \rho(r)}{v_c^2} \left[\mathrm{erf}(1) - \frac{2X}{\sqrt{\pi}} e^1 \right] \\ &= -0.428 \ln \Lambda \frac{GM^2}{r^2}. \end{aligned} \tag{43}$$

Considering that this force is tangential to the trajectory of the galaxy one can write the change of angular momentum per unit mass as

$$\frac{dL}{dt} = \frac{F_{df} r}{M} \simeq -0.428 \frac{GM}{v_c} \ln \Lambda, \tag{44}$$

and since for an isothermal sphere $L = r v_c$ one derives the following equation of motion for the galaxy:

$$r \frac{dr}{dt} = -0.428 \frac{GM}{v_c} \ln \Lambda. \tag{45}$$

6.2 Collisional Dynamics: Shock Heating & Radiative Cooling

Part of the gas (depending on the mass of the halo) which is captured in dark matter halos will be shock heated during its infall, thereby acquiring a characteristic temperature: the virial temperature T_{vir}. To estimate it, one usually assumes that the shock-heated gas is in hydrostatic equilibrium within the potential well created by the dark matter halo, i.e. it has sufficient pressure support to stop collapsing. However, it is possible to get a good approximation by using simple thermodynamic relations. Let us consider how the kinetic energy of the gas and its thermal energy relate:

$$\frac{1}{2}\mu m_p \sigma^2 = \frac{3}{2}kT, \tag{46}$$

where μ is the average mass of the gas in units of the proton mass m_p, σ is the velocity dispersion of the gas and k is the Boltzmann constant. One can then derive an expression for T_{vir} simply by rearranging Eq. 46:

$$T_{vir} = \frac{\mu m_p \sigma^2}{3k}. \tag{47}$$

It is convenient to relate the virial temperature to another measurable quantity which is the circular velocity V_c. This velocity is the velocity needed for a test-particle to be on a circular orbit around a mass distribution. Of course this value depends on the density profile of the halo/galaxy. In the case of the singular isothermal sphere Eq. 27 the relation between velocity dispersion and circular velocity is $V_c^2 = 2\sigma^2$, and is independent of radius, which is the reason why the profile is called *isothermal*. Note that in this very same case of an isothermal density profile calculations based on hydrostatic equilibrium yield:

$$T_{vir} = \frac{1}{2}\frac{\mu m_p}{k}V_c^2 = 35.9\left(\frac{V_c}{\text{km/s}}\right)^2 \text{Kelvin} \tag{48}$$

Gas of a given temperature emit photons which carry away the thermal energy of the gas: this process is called radiative cooling. In galaxies/groups/clusters, this happens mainly through collisions between atoms, during which these atoms get excited/ionized and hence radiate away photons when they return to an unexcited state and/or when the ions capture back their electrons. These processes are efficient down to temperature of around $T \sim 10^4$K, and it is worth noting that at temperatures above 10^6 K the main source of cooling is radiation from Bremsstrahlung.

One can estimate the time it will take for gas at a given temperature to cool down completely, i.e. radiate away all of its thermal energy:

$$t_{cool}(r) = \frac{E}{\dot{E}} = \frac{3}{2}\frac{NkT}{n_e n_i \Lambda(Z,T)}, \tag{49}$$

where E is the total thermal energy density of the gas, N the number density of particles in the gas, n_e the number density of electrons, n_i the number density of ions and $\Lambda(Z,T)$ is the *cooling function* of the gas which depends on the temperature and the metallicity, $Z \equiv m_{heavyelements}/m_{gas}$, of the gas. The cooling function is the net sum of all possible cooling channels for the gas (all the possible transitions of atoms and ions) in units of erg $\mathrm{cm}^3\ \mathrm{s}^{-1}$ and, in general, needs to be numerically calculated using quantum mechanics. The density of the gas is given by

$$\rho = N\mu m_p \tag{50}$$

where N is the number density of particles

$$N = n_e + \sum_i n_i \ , \tag{51}$$

μ is the average mass of particles in the gas and m_p is the proton mass. Using these relations Eq. 49 can be rewritten as

$$t_{cool}(r) = \frac{3}{2}\frac{\rho(r)kT}{\mu m_p n_e n_i \Lambda(Z,T)} \tag{52}$$

in the most general case, and defines the cooling time t_{cool} as the time it takes the gas to radiate away all its thermal energy. For a hydrogen rich gas one can simplify Eq. 52 since $n_e n_i = n_e^2$ and $N = 2n_e$, which leads to the following expression for the cooling time:

$$t_{cool}(r) = \frac{6\mu m_p kT}{\rho(r)\Lambda(Z,T)}. \tag{53}$$

As Eq. 53 shows, and one would intuitively expect, cooling is more efficient, i.e. the cooling time is shorter, when the density of the gas is larger.

6.3 Gravitational Lensing

Gravity bends the path of photons. This makes gravitational lensing an extremely powerful tool to study the mass distribution of matter in the Universe, since this effect is independent of whether or not clusters or groups have reached an equilibrium or are still growing and changing.

6.3.1 Lensing by a compact object: microlensing

The Sun bending of light rays was the first test of Einstein's theory of general relativity. If the gravity of a mass M, located at point L, bends the light of a source located at point S' in the image plane — this plane is perpendicular to the line $O - L$ it defines with and observer located at point O —, then this source will appear to be located at point I in the image plane instead of S'. Einstein predicted that the light passing at a distance b from L in the lens plane is bent by an angle α approximately given by:

$$\alpha \approx \frac{4GM}{bc^2} = \frac{2R_s}{b} \tag{54}$$

where G is the gravitational constant, c the speed of light and $R_s = 2GM/c^2$ is tha Schwarschild radius beyond which the light cannot escape from the gravitational pull of the mass. This approximation holds as long as the bending is small, i.e. $\alpha \ll 1$.

Using Eq.54 we can now calculate where the image of a distant source will appear if a point mass is placed in front of it. If the mass lens had been absent, we would have seen the source at S' at an angle $\beta = LOS'$ from the direction $O - L$. Writing d_S as the distance from the observer to the image plane (i.e. the intersection S of $O - L$ with the plane containing S') and y the distance $S - S'$ in the plane, we have $\beta \approx y/d_S$ as long as $d_S \gg y$. Because the light is bent by an amount α, the source appears at an angle $\theta = LOI$ instead of β. If we call x the distance $S - I$ in the image plane, we have $\theta \approx x/d_S$ as long as $d_S \gg x$. For a small bending, the displacement of the point mass in the image plane is then given by $x - y = \alpha d_{LS}$ where d_{LS} is the distance between the lens at L and the image plane, i.e. $L - S$. Finally, the impact parameter b, which measures the distance between the intersection L' of a light ray emitted by the source and the lens plane (i.e. the plane perpendicular to $O - L$ and containing L), simply is $b = \theta d_L$ if $d_L \gg b$ and d_L is the distance between the observer and the lens $O - L$. Using Eq.54 and dividing it by d_S we find:

$$\beta - \theta = \alpha \frac{d_{LS}}{d_S} = \frac{1}{\theta}\frac{4GM}{c^2}\frac{d_{LS}}{d_L d_S} \equiv \frac{1}{\theta}\theta_E^2 \qquad (55)$$

where the angle θ_E is called the *Einstein radius.* We then have to solve a quadratic equation to find the angular distance θ between the lens L and the image of the point source L' (or I since we are only interested in the angular separation here):

$$\beta^2 - \beta\theta - \theta_E^2 = 0 \text{ i.e. } \theta = \frac{\beta \pm \sqrt{\beta^2 + 4\theta_E^2}}{2} \qquad (56)$$

We can see immediately from Eq.56 that a point source located exactly behind the lens (i.e. at point S) will be seen as a circle of light on the sky with radius θ_E since $\theta = 0$ in this case. When $\theta > 0$, the image θ_+ is further from the lens since $\theta_+ > \beta$ and lies outside the Einstein radius since $\theta_+ > \theta_E$. These exterior images were the ones seen around the eclipsed Sun by Eddington. The image at θ_- is inverted and lies within θ_E on the opposite side of the lens.

Quite regularly, one star of the Milky Way's bulge is gravitationally lensed by another one in the disk. However, images θ_+ and θ_- are too close from one another to distinguish individually in that case, but one can still tell the star is beeing lensed as it appears brighter on the sky. Because of the small size of the Einstein ring, gravitational lensing by compact objects in the halo is often called *microlensing.*

6.3.2 Lensing by extended sources: galaxies and clusters

When the lens is an entire galaxy or a cluster of galaxies, we can first think of it as a collection of point masses. We then rewrite Eq.54 so as to define a *lensing potential* Ψ_L:

$$\alpha(b) \equiv \frac{d\Psi_L}{db} \text{ with } \Psi_L(b) = \frac{4GM}{c^2}\ln b \qquad (57)$$

We can now calculate the bending of the light emitted by a background source by summing up the effects of all the point masses within the lens.

If the lens is compact as compared with both its distance d_L to the observer **and** its distance d_{LS} to the source plane, then the deflection of the light only depends on the *continuous* surface density $\Sigma(\vec{x})$ of the lens. In this continuous limit, one must then specify the light ray's closest approach to the galaxy/cluster center by a vector $\vec{b}$ and integrate over the galaxy/cluster to calculate the deflection vector $\vec{\alpha}$:

$$\vec{\alpha}(\vec{b}) \equiv \nabla\Psi_L(\vec{b}) \text{ with } \Psi_L(\vec{b}) = \frac{4GM}{c^2}\int_S \Sigma(\vec{b'}) \ln|\vec{b}-\vec{b'}|d^2b' \tag{58}$$

where the lensing potential has a similar form to the gravitational potential $\phi(\vec{b})$ defined in equation 9 but the integral is two dimensional and the term $1/|\vec{b}-\vec{b'}|$ is replaced by the $\ln|\vec{b}-\vec{b'}|$ term. In the most general of cases, we must calculate $\Psi_L(\vec{b})$ numerically with a computer from a distribution of matter given by $\Sigma(\vec{b})$, but suppose that the lens is axisymmetric, so that Σ only depends on the projected distance R of the source to the center of the galaxy/cluster. We can then show that the bending of a ray of light passing at radius b from the center of the lens only depends on the mass $M_{<b}$ **projected** within that circle. Eq. 58 then simplifies to:

$$\alpha(b) = \frac{4G}{bc^2}\int_0^b \Sigma(R)2\pi R dR = \frac{4G}{c^2}\frac{M_{<b}}{b}. \tag{59}$$

To prove this, we can just adapt the arguments that we used to prove Newton's second theorem [5], so that the light is bent just as if all the material projected within radius b had been replaced by a point of the same total mass located at the center.

Let us now use Eq. 59 to figure out how an axisymmetric galaxy/cluster bends the light emitted by a distant galaxy located behind it. The geometry is the same as for micro lensing, except that the lens is now not a compact object! If the lensing cluster had been absent, we would have seen the background galaxy at point S', i.e. at an angle β from $O-L$, the line which joins the observer to the cluster center. Instead, we see this background galaxy's image at point I which makes an angle θ with $O-L$. Thus we only have to modify Eq. 55 which was valid for microlensing in the following way:

$$\beta - \theta = \alpha\frac{d_{LS}}{d_S} = \frac{1}{\theta}\frac{4GM_{<b}}{c^2}\frac{d_{LS}}{d_L d_S}, \tag{60}$$

remembering that $b = \theta d_L$. We can rewrite Eq.60 in terms of the *critical density for lensing* Σ_{crit}:

$$\beta = \theta\left[1 - \frac{1}{\Sigma_{crit}}\frac{M_{<b}}{\pi b^2}\right] \text{ where } \Sigma_{crit} \equiv \frac{c^2}{4\pi G}\frac{d_S}{d_L d_{LS}}. \tag{61}$$

The quantity $M_{<b}/(\pi b^2)$ is simply the average surface density within radius b. Usually, $\Sigma(R)$ declines from a peak at the center of the lens, so that this average will fall as well. It naturally follows that if the central density is greater than Σ_{crit}, then the image of a source located at $\beta = 0$ exactly in line with the observer and the cluster center will be a thin circular *Einstein ring* of angular size $\theta_E = b_E/d_L$ where b_E is the radius where the average density equals the critical density value, i.e. $M_{<b_E}/(\pi b_E^2) = \Sigma_{crit}$. On the other hand, if the central surface density of the lens is smaller than Σ_{crit}, then the cluster cannot produce multiple images of any background galaxy and no ring is observed.

[5] We can also prove in the same way the equivalent of Newton's first theorem: a light ray passing through a uniform circular ring is not bent.

6.3.3 Weak gravitational lensing

When galaxies lie behind a lensing cluster, but are located well outside of its Einstein radius, their images are only weakly magnified and slightly stretched in the tangential direction, i.e. galaxies which would otherwise appear as perfect circular discs become ellipses with tangential and radial axes having a ratio $\frac{x}{y}/\frac{\Delta x}{\Delta y}$ or $|\frac{d\beta}{d\theta}|/|\frac{\beta}{\theta}|$ when using the same notation as in the microlensing case.

The *shear* γ is therefore used in weak lensing studies, which measures the difference in the amount of compression the lens exerts on the source in the tangential and the radial directions. For an image located at a distance $b \gg \theta_E d_L$ from the lensing cluster's center, we have:

$$\gamma \equiv \frac{1}{2}\left[\frac{d\beta}{d\theta} - \frac{\beta}{\theta}\right] = \frac{\bar{\Sigma}_{<b} - \Sigma(b)}{\Sigma_{crit}} \tag{62}$$

where $\bar{\Sigma}_{<b} = M_{<b}/(\pi b^2)$ is the average surface density of matter projected within radius b and $\Sigma(b)$ is the surface density at radius b.

Measuring the average shape of many background galaxies that have been weakly distorted allows one to estimate the shear and hence probe the (projected) mass distribution in the outer parts of galaxy clusters. Note that this technique also applies to distortions caused by the large scale structure of the Universe, but in this case the light from background galaxies is bent multiple times by each of the structure present on its way to the observer, so one cannot assume the bending occurs at a single distance d_L. The only way to calculate the amplitude of the weak lensing shear signal originating from large scale structure lensing is therefore to perform ray-tracing numerical simulations by post-processing the matter density distribution provided by a cosmological simulation.