# Astrophysics ($4^{\rm th}$ year): 5 Lectures on the Large Scale Structure of the Universe

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## Preamble

These are a set of short notes which accompany the 5 lectures that I give within the $4^{th}$ year Astrophysics course. They are an almost direct transcription of what I write on the board and therefore cannot be considered a complete text on the subject, merely a guide. All class problems and exam questions will be based on the material I present in class (and therefore in these notes). I apologize for the multiple typos.

I have followed quite a few books on cosmology and galaxies to prepare these lectures. These are

[L]: A. Liddle, *An Introduction to Modern Cosmology*, Wiley

[P]: J. Peacock, *Cosmological Physics*, CUP

[D]: S. Dodelson, *Modern Cosmology*, Academic Press

[KT]: E.Kolb & M. Turner, *The Early Universe*, Addison Wesley.

In each section I will suggest where to best study any given topic. Note that there are a number of good websites on cosmology. Just pick a topic and google it.

# 1 Newtonian Evolution Equations of Cosmological Perturbations

Our current understanding of the expanding Universe is based an extraordinary simplifying assumption- that at any given time, it looks the same everywhere and any direction we may wish to look. This assumption of homogeneity and isotropy is borne out by our observations of the cosmic microwave background which we find to be isotropic to within one part in a hundred thousand. Yet we know that the observable is remarkably smooth and isotropic but it is not perfectly so. We see a plethora of structures, from clusters, filaments and walls of galaxies to large empty voids that can span hundreds of millions of light years. Indeed, the fact that there are galaxies, stars and planets indicate that the Universe is not at all smooth as we observed it on smaller and smaller scales. Hence to have a complete understanding of the dynamics and state of the Universe and to be able to accurately predict its large scale structure, we go beyond describing it just in terms of a scale factor, overall temperature., density and pressure.

If we are to explore departures from homogeneity, we must study the evolution of energy density, $\rho$, pressure, $P$ and gravity, $\Phi$ in an expanding universe in a more general context, allowing for spatial variations in these various contexts. We will restrict ourselves to Newtonian gravity which will give us the qualitative and quantitative behaviour of perturbations that we would find in a proper, general relativistic treatment. Let us focus on the evolution of pressureless matter, appropriate for the case of massive, non-relativistic particles[1]. .

The evolution of a gravitating pressureless fluid is governed by a set of conservation equations known as the Euler equations. We have that *conservation of energy* is given by

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0 \tag{1}$$

while *conservation of momentum* is given by

$$\frac{\partial V}{\partial t} + (\vec{V} \cdot \nabla)\vec{V} = -\nabla\Phi - \frac{1}{\rho}\nabla P \tag{2}$$

Note that we have had to introduce the fluid velocity, $\vec{V}$ into our system. These conservation equations are complemented by the Newton-Poisson equation

$$\nabla^2 \Phi = 4\pi G \rho \tag{3}$$

[1]It is possible to generalize the Euler equations to a relativistic fluid on Minkwoski space. They will be replaced by a conservation of energy equation of the form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot [(\rho + P)\vec{V}] = 0,$$

a conservation of momentum equation of the form

$$\frac{\partial V}{\partial t} + (\vec{V} \cdot \nabla)\vec{V} = -\frac{\rho}{\rho + P}\nabla\Phi - \frac{1}{\rho + P}\nabla P$$

and modified Newton-Poisson equation of the form

$$\nabla^2 \Phi = 4\pi G(\rho + 3P)$$

which gives us how this system behaves under gravity.

We can clearly see that these set of equations are strictly valid for a universe dominated by pressurless matter if we attempt to solve for the mean density $\rho_0$ and mean expansion $\vec{V}_0 = H\vec{r}$ corresponding to a homogeneous and isotropic universe. Solving Equation 1 we have that

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho_0 \vec{V}_0) = -\rho \nabla \cdot \vec{V}_0 = -3H\rho$$

which gives us $\rho \propto a^{-3}$.

The Euler equations are, in general, difficult to solve. We can, however, study what happens when the Universe is mildly inhomogenous, i.e. we can consider small perturbations around these values so that the total density, pressure, velocity and gravitational potential at a given point in space can be written as $\rho = \rho_0 + \delta\rho$, $P = P_0 + \delta P$, $\vec{V} = \vec{V}_0 + \delta\vec{v}$, $\Phi = \Phi_0 + \delta\Phi$, where $\delta\rho/\rho \ll 1$, $\delta P/P \ll 1$ and so one. This approach is known as *Cosmological Perturbation Theory*- it involves study of small perturbations to a FRW universe and we will find that the evolution equations greatly simplify in this regime. We can first start off with the conservation of energy equation

$$\frac{\partial(\rho_0 + \delta\rho)}{\partial t} + \nabla \cdot [(\rho_0 + \delta\rho)(\vec{V}_0 + \delta\vec{v})] = 0$$

which we can expand to give us:

$$\frac{\partial \rho_0}{\partial t} + \nabla \cdot (\rho_0 \vec{V}_0) + \frac{\partial \delta\rho}{\partial t} + \nabla \cdot (\rho_0 \delta\vec{v}) + \nabla \cdot (\delta\rho \vec{V}_0) + \nabla \cdot (\delta\rho\delta\vec{v}) = 0$$

The first two terms satisfy the conservation equations as seen above while we the last term is a product of two very small quantities and hence is negligible. It is possible to further simplify the equations using $\nabla \cdot \vec{V}_0 = 3H$ and defining $\delta \equiv \delta\rho/\rho_0$. If we convert the partial derivative in time to a total time derivative

$$\frac{d\delta}{dt} = \frac{\partial \delta}{\partial t} + \vec{V}_0 \cdot \nabla\delta$$

we then find that the first order conservation of energy equation reduces to

$$\frac{d\delta}{dt} + \nabla \cdot \delta\vec{v} = 0 \tag{4}$$

The same can be done to the conservation of momentum equation,

$$\frac{d\delta\vec{v}}{dt} + H\delta\vec{v} \;=\; -c_S^2 \nabla\delta - \nabla\delta\Phi \tag{5}$$

where we have defined the speed of sound of this fluid to be $c_s^2 = \frac{\nabla\delta P}{\nabla\delta\rho}$ and the perturbed Newton-Poisson equation[2] becomes

$$\nabla^2\Phi = 4\pi G\rho_0\delta \tag{6}$$

[2]In a homogeneous and isotropic Universe we can assume that $\Phi_0 = 0$

The system has now been simplified to a set of linear differential equations with time dependent coefficients which can be solved either numerically or approximately using Fourier transforms.

There is a further transformation we can do to simplify the system. First of all it is important to note that we have been working in physical coordinates, $\vec{r}$, and that it is much more convenient to switch to conformal coordinates, $\vec{x}$ (i.e. coordinates that are defined on the space-time grid); we then have $\vec{r} = a\vec{x}$ so that gradients between the two coordinate systems are related through $\nabla_r = \frac{1}{a}\nabla_x$ and the velocity perturbations are related through $\delta\vec{v} = a\vec{u}$. If we make a further simplifying assumption that there are no vortical flows in the fluid, we can define a new variable $\Theta = \nabla \cdot \vec{u}$. We then have that Equation 5 becomes

$$\dot{\Theta} + 2H\Theta = -\frac{c_s^2}{a^2}\nabla^2\delta - 4\pi G\rho_0\delta$$

Combined with Equation 4 we can rewrite the perturbed Euler equation as a $2^{\rm nd}$ order linear partial differential equation $\delta$:

$$\ddot{\delta} + 2H\dot{\delta} - \frac{c_s^2}{a^2}\nabla^2\delta = 4\pi G\rho_0\delta \tag{7}$$

We have derived a partial differential equation for $\delta$ which is more easily studied and solved in Fourier space. If we take the Fourier transform[3] , $\delta \to \delta_{\mathbf{k}}$, Equation 7 becomes

$$\ddot{\delta}_k + 2H\dot{\delta}_k = (-\frac{c_s^2}{a^2}k^2 + 4\pi G\rho_0)\delta \tag{8}$$

## 2 Relativistic Cosmological Perturbation Theory

In the previous section we used Newtonian gravity to find an equation that describes how over densities in the mass distribution evolve. We know, however, that the correct theory of gravity is General Relativity and it would make sense to derive a set of equations for how inhomogeneities evolve which is completely consistent with how we derived the evolution of the Universe as a whole. Again, we shall apply the rules of linear perturbation theory but now we do so to the Einstein field equations and the relativistic conservation of energy and momentum. We won't push the calculation to its bitter end but will give you a flavour of all the steps involved. For a complete analysis, see [D].

We will be working with the perturbed cosmological metric (and we assume $c = 1$):

$$ds^2 = -(1+2\Psi)dt^2 + a^2(1+2\Phi)d\vec{x}^2$$

Note that different textbooks/papers us different convention. Also note that we are making a specific *gauge* choice here- see [D] for a discussion of what this means. You are familiar with the Newtonian limit where you take $a = 1$ and $\Psi = -\Phi$. We can think of the metric consisting of a zeroth order part (which is just the usual FRW metric) and a linearly perturbed part.

[3] The Fourier transform is taken to be

$$\delta(t,\vec{x}) = \frac{1}{(2\pi)^{3/2}}\int d^3k\delta_{\mathbf{k}}\exp(-i\vec{k}\cdot\vec{x})$$

We now want to work out the linearly perturbed Einstein field equations which will look schematically like

$$\delta G_{\alpha\beta} = 8\pi G \delta T_{\alpha\beta}$$

So we need to work out both sides of this equation. Let us start with the left hand side, and to do so, we need to work out the connection coefficients, $\Gamma^{\alpha}{}_{\mu\nu}$, the Ricci tensor, $R_{\alpha\beta}$ and the Ricci scalar $R$, find the zeroth order parts and the linearly perturbed parts.

To simplify, we can start with

$$\Gamma^{0}{}_{\mu\nu} = \frac{1}{2} g^{0\alpha}(\partial_\nu g_{\mu\alpha} + \partial_\mu g_{\alpha\nu} + \partial_\alpha g_{\mu\nu})$$

Note that to get $g^{\alpha\beta}$ we use the fact that, for small $\epsilon$, $(1+\epsilon)^{-1} \simeq 1-\epsilon$. We then see that only $\alpha = 0$ matters and we find that (discarding all terms of order 2- i.e. terms which are quadratic or higher order in small quantities);

$$\begin{aligned}
\Gamma^{0}{}_{00} &= -\frac{1-2\Psi}{2}(-2\partial_0\Psi) = \partial_0\Psi \\
\Gamma^{0}{}_{0i} &= -\frac{1-2\Psi}{2}(-2\partial_i\Psi) = -ik_i\Psi \\
\Gamma^{0}{}_{ij} &= a^2\delta_{ij}[H + 2H(\Phi-\Psi) + \partial_0\Phi]
\end{aligned}$$

Note a few things. First of all, we have replaced $\partial_i \to -ik_i$. Second, the last expression has a zeroth order bit and first order bit- the zeroth order bit you have already derived when working out the FRW equations from scratch in General Relativity. In fact, if you remember, the only non zero connection coefficients for the FRW metric are $\Gamma^{0}{}_{ij}$ and $\Gamma^{i}{}_{0j}$. If you now work out the remaining connection coefficients, you find:

$$\begin{aligned}
\Gamma^{i}{}_{00} &= \frac{-ik_i\Psi}{a^2} \\
\Gamma^{i}{}_{j0} &= \delta_{ij}(H + \partial_0\Phi) \\
\Gamma^{i}{}_{jk} &= -i\Phi[\delta_{ij}k_k + \delta_{ik}j_k - \delta_{jk}k_i]
\end{aligned}$$

Now we turn to the Ricci scalar. Again, let us focus on one:

$$R_{00} = \partial_\alpha\Gamma^{\alpha}{}_{00} - \partial_0\Gamma^{\alpha}{}_{0\alpha} + \Gamma^{\alpha}{}_{\beta\alpha}\Gamma^{\beta}{}_{00} - \Gamma^{\alpha}{}_{\beta 0}\Gamma^{\beta}{}_{0\alpha}$$

If $\alpha = 0$, all terms cancel so we only need to sum over $\alpha = i$. We then have four terms:

$$\begin{aligned}
\partial_i\Gamma^{i}{}_{00} &= -\frac{k^2}{a^2}\Psi \\
-\partial_0\Gamma^{i}{}_{0i} &= -3\left(\frac{d^2a/dt^2}{a} - H^2 + \partial_0^2\Phi\right) \\
\Gamma^{i}{}_{\beta i}\Gamma^{\beta}{}_{00} &= 3H\partial_0\Psi \\
-\Gamma^{i}{}_{\beta 0}\Gamma^{\beta}{}_{0i} &= -3(H^2 + 2H\partial_0\Phi)
\end{aligned}$$

So we find

$$R_{00} = -3\frac{d^2a/dt^2}{a} - \frac{k^2}{a^2}\Psi - 3\partial_0^2\Phi + 3H(\partial_0\Psi - 2\partial_0\Phi)$$

Note that, once again, it splits into a zeroth order bit and a linear bit. We can repeat this for the $i$ and $j$ components of the Ricci tensor to find

$$\begin{aligned} R_{ij} &= \delta_{ij}[(2a^2H^2 + a\frac{d^2a}{dt^2})(1 + 2\Phi - 2\Psi) + \\ &\quad a^2H(6\partial_0\Phi - \partial_0\Psi) + a^2\partial_0^2\Phi + k^2\Phi] + k_ik_j(\Phi + \Psi) \end{aligned}$$

Finally, we can find the Ricci scalar

$$\begin{aligned} R = 6(H^2 + \frac{d^2a/dt^2}{a}) &- 12(H^2 + \frac{d^2a/dt^2}{a})\Psi + \frac{2k^2}{a^2}\Psi + 6\partial_0^2\Phi \\ &- 6H(\partial_0\Psi - 4\partial_0\Phi) + \frac{4k^2\Phi}{a^2} \end{aligned}$$

We now have all the ingredients to work out the left hand side of the Einstein field equations. Again, let us focus on the 00 component. We have that

$$\delta G^0{}_0 = \delta R^0{}_0 - \frac{1}{2}\delta R\delta^0{}_0 = \delta(g^{0\mu}R_{\mu 0}) - \frac{1}{2}\delta R\delta^0{}_0$$

Be careful not confuse the Kronecker delta with the symbol of linear perturbation. We find that

$$\delta G^0{}_0 = -6H\partial_0\Phi + 6H^2\Psi - 2\frac{k^2}{a^2}\Phi$$

For the right hand side we need to perturb the energy momentum tensor. Let us restrict ourselves to a perfect fluid for which we know that

$$T^0{}_0 = (\rho + P)U^0U_0 + P\delta^0{}_0$$

with $U_\mu U^\mu = -1$. We then have that $\delta T^0{}_0 = -\delta\rho$. If we now combine that with the expression for $\delta G^0_0$ we end up with

$$\frac{k^2}{a^2}\Phi + 3H(\partial_0\Phi - H\Psi) = 4\pi G\delta\rho$$

If we now transform to conformal time (and define $\mathcal{H} = aH$), we end up with

$$k^2\Phi + 3\mathcal{H}(\Phi' - \mathcal{H}\Psi) = 4\pi Ga^2\delta\rho.$$

This equation should look mildly familiar. Let us make it look even more familiar.

We can use the same process as we used above to show that

$$\delta G^i{}_j = A\delta^i_j + \frac{k_ik_j(\Phi + \Psi)}{a^2}$$

where $A$ is a long expression built up out of $\Phi$, $\Psi$ and their derivatives. If we perturb $T^i_{\ j}$ we find that

$$\delta T^i_{\ j} = \delta P \delta^i_{\ j}$$

The latter is true because we are only perturbing $\rho$, $P$ and $U^\mu$. Now let us contract the perturbed Einstein field equation

$$\delta G^i_{\ j} = 8\pi G \delta T^i_{\ j}$$

with the tensor

$$\mathcal{P}^i_{\ j} = k^i k_j - \frac{1}{3} k^2 \delta^i_{\ j}$$

You quickly find that $\mathcal{P}^i_{\ j}$ contracted with $\delta^i_{\ j}$ is zero to leave

$$\frac{2k^2}{3a^2}(\Phi + \Psi) = 0$$

That is, the Einstein Field equations in the absence of anisotropic stresses set $\Phi = -\Psi$. If we now plug that back into the 00 equation we find

$$-k^2\Psi - 3\mathcal{H}(\Psi' - \mathcal{H}\Psi) = 4\pi G a^2 \delta\rho.$$

which, for $a = 1$ gives us the Newton-Poisson equation. If we assume $\Psi' \simeq \mathcal{H}\Psi$ and recall that $\mathcal{H} \simeq 1/\eta$ we have that the Newton Poisson equation is a good approximation when $k\eta \gg 1$, that is when the wavelengths of the perturbations are much smaller than the horizon.

The "relativistic" Newton-Poisson equation can be written in another form if we take $\delta G^0_{\ i} = 8\pi G \delta T^0_{\ i}$. Taking the perturbed velocity 4-vector, $U^\mu = (1, \vec{v})$ and defining $\theta = \vec{\nabla} \cdot \vec{v}$ and we can combine two components of the Einstein field equation so that

$$\frac{1}{2}\left[G^0_{\ 0} - i\frac{3\mathcal{H}}{k^2} k^i \delta G^0_{\ i}\right] = -k^2\Psi = 4\pi G a^2 \rho_0 \left(\delta - \frac{3\mathcal{H}}{k^2}\theta\right)$$

Again, note the correction to the Newton-Poisson equation.

To close the system we now need to use the relativistic equivalent of energy conservation and the Euler equation. This will come from taking the linear term of the covariant conservation of the energy momentum tensor, $\delta(\nabla_\mu T^{\mu\nu} = 0$. If we define the sound speed, $c_s^2 = \delta P/\delta\rho$ and recall that the equation of state is $P = w\rho$, we have that the equations are

$$\begin{aligned} \delta' &= -(1+w)(\theta - 3\Psi') - 3\mathcal{H}(c_s^2 - w)\delta \\ \theta' &= -\mathcal{H}(1-3w)\theta + \frac{c_s^2 k^2}{1+w}\delta + k^2\Psi \end{aligned}$$

All together, these equation can be used to study the evolution of perturbations on all scales for a range of $w$ and $c_s^2$ and $\mathcal{H}$.

# 3 The evolution of large scale structure

A cursory glance at Equation 8 allows us to identify a number of features in the evolution of $\delta$ without actually solving the system. For a start, it is quite clearly the equation for a damped harmonic oscillator with time dependent damping coefficient and spring constant. The damping is due to the expansion of the Universe and will tend to supress growth. The spring constant will change sign depending on whether $k$ is large or small. If the positive part of the spring constant, $c_s^2k^2/a^2$, dominates then we should expect oscillatory behaviour in the form of acoustic waves in the fluid. If the negative term, $4\pi G\rho_0$ dominates, then the evolution will be unstable and we should expect $\delta$ to grow. The *physical* (as opposed to conformal) wavelength, $\lambda_J$ , that defines the transition between these two behaviours is given by

$$\lambda_J = c_s\left(\frac{\pi}{G\rho_0}\right)^{\frac{1}{2}}$$

and is known as the *Jeans wavelength.* For $\lambda > \lambda_J$ gravitational collapse dominates and perturbations grow. For $\lambda < \lambda_J$ pressure will win out and perturbations will not grow. We can have a rough idea of how a given system of particles will behave if we note that $c_s^2 \sim (K_BT)/(Mc^2)$ where $T$ is the temperature of the system and $M$ is the mass of the individual particles. We can then rewrite the Jean length as

$$\lambda_J = \left(\frac{\pi K_BT}{GM\rho_0}\right)^{1/2}$$

It is clear that a hot system or made up of light particles will have a large $\lambda_J$; a cold system with heavy particles will have a small $\lambda_J$.

It is often convenient to write the evolution equation for density perturbations in terms of conformal time $\eta$; recall that $dt = ad\eta$ so we can solve for $\eta = \int \frac{dt}{a}$ and we now denote $X' = \frac{dX}{d\eta}$. Equation 8 now becomes

$$\delta_k'' + \frac{a'}{a}\delta_k' + (c_s^2k^2 - 4\pi G\rho a^2)\delta_k = 0 \qquad (9)$$

It is now useful to find solutions for specific scenarios.

## 3.1 Pressureless fluid in the matter dominated era.

In this situation we have that $c_s^2 \simeq 0$ and hence $\lambda \gg \lambda_J$. We can therefore discard the term which depends on pressure to get:

$$\ddot{\delta} + 2H\dot{\delta} - \frac{3}{2}H^2\delta = 0$$

(we have used $\frac{3}{2}H^2 = 4\pi G\rho_0$. From $a = (t/t_0)^{2/3}$ we have $H = 2/3t$ and the solutions are then

$$\delta = C_1t^{\frac{2}{3}} + C_2t^{-1} \qquad (10)$$

The second term decays and becomes subdominant very fast and we are left with the first term which can be rewritten as $\delta \sim a$. If repeat the same calculation now using conformal time we find $\delta_k \propto \eta^2$ and $\eta^{-3}$. Hence we find that in this situation, perturbations grow due to the effect of gravity; the growing solution, $\delta_k \sim a$ is normally called the *growing mode.*

### 3.2 Pressureless matter in the $\Lambda$ dominated era.

The growth rate of perturbations will depend on the expansion rate of the Universe. If we wish to study the growth rate during a period in which the cosmological constant dominates, we have that

$$H^2 = \Lambda/3$$

and therefore $a \propto \exp(Ht)$. We also have that Equation 6 will be modified to include two components:

$$\nabla^2\Phi = 4\pi G(\rho_\Lambda\delta_\Lambda + \rho_M)\delta_M = 4\pi G\rho_\Lambda(\delta_\Lambda + \frac{\rho_M}{\rho_\Lambda}\delta_M) = \frac{3}{2}H^2(\delta_\Lambda + \frac{\rho_M}{\rho_\Lambda}\delta_M)$$

The energy density of the cosmological constant (which is constant in space, i.e. $\delta_\Lambda = 0$) dominates so $\rho_M \ll \rho_\Lambda$ and we find

$$\ddot{\delta} + 2H\dot{\delta} = \frac{3}{2}\frac{\rho_M}{\rho_\Lambda}\delta_M \simeq 0$$

where $H$ is a constant. The solutions are $\delta \sim \exp(-Ht)$ and $\sim$ constant. Clearly a faster rate of expansion inhibits growth.

### 3.3 Relativistic fluid in the radiation dominated era.

The characteristic properties of the fluid will also affect how it evolves under gravity in an expanding universe. Consider the growth of perturbations of a relativistic fluid with pressure. An example of this scenario is of radiation interacting strongly with baryons before recombination. During this epoch, baryons are dissociated into protons and electrons which interact with photons through Thomson scattering. The net result is that radiation behaves as a gravitating fluid with pressure. The sound speed is $c_S^2 = 1/3$. As was mentioned above, the Euler equations are modified in the case of a relativistic fluid and pressure will play a significant role. Equation 8 is now replaced by

$$\ddot{\delta}_k + 2H\dot{\delta}_k = [-\frac{c_s^2}{a^2}k^2 + \frac{16}{3}\pi G(\rho_0 + 3P_0)]\delta$$

It is easier to solve the corresponding equation in conformal time, where (taking $P_0 = \rho_0/3$) we have

$$\delta_k'' + \frac{a'}{a}\delta_k' + \left[\frac{k^2}{3} - 4\left(\frac{a'}{a}\right)^2\right]\delta_k = 0$$

The scale factor in the radiation era evolves as $a = \sqrt{t/t_0}$ and hence, with $a \simeq \eta$, we find that

$$\delta_k'' + \frac{1}{\eta}\delta_k' + \left(\frac{k^2}{3} - \frac{4}{\eta^2}\right)\delta_k = 0$$

This equation can be solved exactly and we find that a well-behaved solution (such that $\delta_k \to 0$ as $\eta \to 0$), is given by

$$\delta_k = AJ_2(\frac{k\eta}{\sqrt{3}})$$

in which $J_2(y)$ is a Bessel function and $A$ is an arbitrary normalizing constant. From the assymptotic behaviour we can probe the large and small scale behaviour. On large scales, i.e. when $k\eta \ll 1$ we have that $\delta_k \propto \eta^2$, very much like the growing mode in matter during the matter dominated era. On small scales, i.e. for $k\eta \gg 1$ we have that

$$\delta_k \to \sqrt{\frac{3}{k\eta}} \cos\left(\frac{k\eta}{\sqrt{3}} - \frac{5\pi}{4}\right).$$

The solution corresponds to a damped acoustic wave which oscillates and decays slightly over time. The length scale of the transition between the two regimes is given roughly by $\frac{k\lambda_H}{2\pi} \simeq 1$ where $\lambda_H = 2\pi\eta/\sqrt{3}$ is the sound horizon of the fluid- the furthest distance a soundwave can propagate from the Big Bang until $\eta$. The solution neatly sums up the competition between the two main effects: on large scale gravity wins and perturbations tend to grow, on small scales pressure dominates and we have "sound" waves. Clearly the sound speed $c_S^2$ plays a crucial role in the dynamics.

## 3.4 Pressureless matter in the radiation dominated era

An interesting situation arises when we have pressureless matter in the radiation era which is decoupled from the baryons and radiation; a notable example is cold dark matter. In this situation, the pressureless matter will not only play a subdominant role in the expansion of the Universe but will also make a negligible contribution to the Newton-Poisson equation. We then have

$$\nabla^2\Phi = 4\pi G(\rho_\gamma\delta_\gamma + \rho_M\delta_M) \simeq 4\pi G\rho_\gamma\delta_\gamma$$

We have solved for $\delta_\gamma$ and we can replace it in Equation 9 to obtain

$$\delta_k'' + \frac{1}{\eta}\delta_k' \propto \frac{1}{\eta^2}J_2(\frac{k\eta}{\sqrt{3}})$$

Once again, we will have two regimes. On scales larger than the sound horizon, we have that $\delta_k \propto \eta^2$ while on small scales we have that $\delta_k \propto \ln(\eta)$. So even though perturbations in the dominant fluid (the radiation) don't grow, perturbations in a subdominant component of pressureless matter will.

## 3.5 Damping of cosmological perturbations

There are two further situations we should examine which do not fit exactly into the formalism we have been using. These occur when there is imperfect coupling between different fluid elements or when the system cannot be described purely in terms of a density field and one must resort to a distribution function.

### 3.5.1 Damping during recombination

For a brief period during recombination, the mean free path of photons will not be negligible nor will it be infinite. The strength of the interaction between the photons and the baryons is given by $a\sigma_T n_e$ where $n_e$ is the number density of free electrons. As the Universe recombines at around a $z \simeq 1100$, $n_e$ will plummet over a redshift change of $\Delta Z \simeq 80$ and the mean free path will be finite. Because the photons and electrons aren't perfectly coupled, the photons will be able to random walk out of overdensities as they scatter off free electrons. In doing so they will shift matter from over densities to underdensities and damp out perturbations on small scales. The damping scale (known as the Silk damping scale) is approximately given by

$$\lambda_S \simeq 2\pi\sqrt{\frac{\eta}{a\sigma_T n_e}}$$

and fluctuations below this scale will be washed out.

### 3.5.2 Massive neutrinos during any era

Massive neutrinos cannot be described as a fluid- they do not interact with each other and their evolution must be studied using the Boltzmann equation. On very large scales they will tend to cluster just like matter and radiation but on small scales, they will tend to *free-stream* i.e. move relativistically from one region of space to another. This will lead an overall damping effect, wiping out structure on small scales. The damping scale will depend on their mass and is roughly given by

$$\lambda_{FS} \simeq 40\left(\frac{30eV}{M_\nu}\right)Mpc$$

## 4 Building models of large scale structure

Thus far we have studied the evolution of structure in a variety of scenarios and we should, by now, have a qualitative understanding of how cosmological perturbations may evolve. We now need complete this analysis by defining the initial conditions, i.e. the seeds of structure, then characterizing how perturbations of different length scales evolve and and finally identifying how we should ultimately characterize large scale structure today.

Over the decades there have been a plethora of proposals for the initial conditions of structure formation. One set of possibilities is that the Universe started off in a quasi-chaotic initial state and that the thermal initial state smoothed out the large inhomogeneities leaving a residue of fluctuations when then evolved to form structure. Clearly this is not a viable proposal unless we severely modify the nature of the Universe at those early time- as we saw in the previous section, structure on very large scales (larger than the Jean wavelength) will tend to grow under the force of gravity. Furthermore, there is a limit to how much the Universe could homogenize set by causality so it is in fact physically impossible to implement such a simple idea.

We do, however, have a proposal that tends to smooth out the Universe and that changes the causal structure of space-time. Inflation will take a microscopic patch of the Universe which is in thermal equilibrium and is well within the Jeans wavelength at that time, and expands

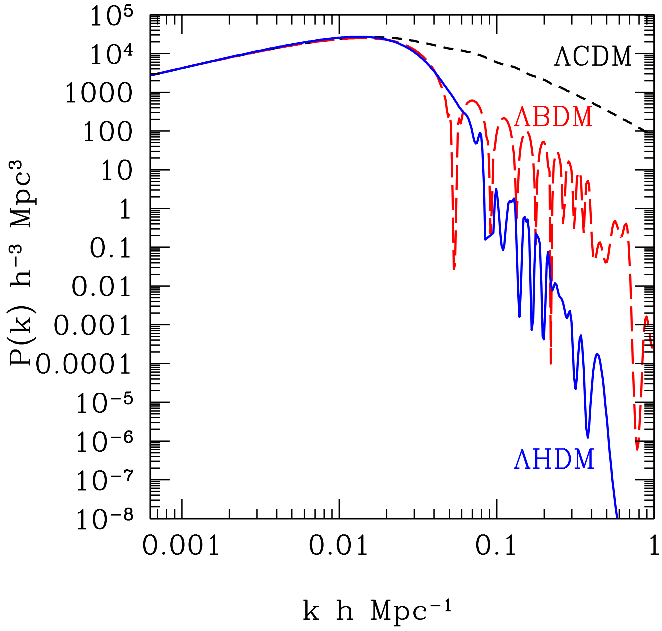


Figure 1: The power spectrum for three different models of the Universe: $\Lambda$ HDM, i.e. a universe with baryons, massive neutrinos and a cosmological constant; $\Lambda$ BDM, i.e. a universe with only baryons and a cosmological constant; $\Lambda$ CDM, i.e. a universe with baryons, cold dark matter and a cosmological constant. Note that the first two models have much less power on small scales (large $k$) than the last one.

it to macroscopic, cosmological proportions. In doing so, Inflation solves the problem of how to homogenize but also provides a mechanism for seeding structure. We expect that, due to the quantum nature of space-time and matter, that the Universe will be riven by quantum fluctuations on microscopic scales. A period of Inflationary expansion will amplify and stretch these quantum fluctuations to macroscopic scales which will be many times larger than the cosmological horizon by the time Inflation ends. As the Universe resumes its normal expansion in the radiation era, the fluctuations will seed structure in the cosmological fluid which will then evolve in the manner described in the previous sections.

The form of the intial conditions arising from Inflation have a deeply appealing feature: they will correspond to a *random field* which has a *scale invariant* gravitational potential. In this context, a random field is a three-dimensional function which can be generated through some random process; this should not come as a surprise given that the source of the fluctuations is a quantum process. And if you think about what we are trying to do, and look at the structure of the sky, you will realized that there must be an element of randomness. Our theory won't tell us if a cluster of galaxies, or a filament of galaxies or more generally an overdensity or underdensity is going to be at an exact position in space. All we can talk about is how much more probable structures of a given size are going to be relative to others. For example, we may expect to see more structure of 1 Mpc than of 100 Mpc, but we don't know exactly where they will be. Hence we talk about our density contrast, $\delta$, or gravitational potentials being a random fields for which we can calculate their statistical properties.

We characterize a random field in much the same way we would characterize any other random process. For example we will have that the density contrast, $\delta$ satisfies

$$\langle\delta(\vec{x})\rangle = 0$$

and that we can characterize its variance in terms of a *correlation function*, $\xi(r)$ through

$$\xi(|\vec{x}-\vec{x}'|) \equiv \langle\delta(\vec{x})\delta(\vec{x}')\rangle$$

or alternatively in terms of its *power spectrum*

$$P(k) \equiv \langle|\delta_{\vec{k}}|^2\rangle$$

By defining $\xi(r)$ or $P(k)$ we can characterize the statistical properties of the random field[4] It is often useful to consider the dimensionless version of the power spectrum, the *mass variance* which is given by

$$\Delta^2(k) = \frac{k^3P(k)}{4\pi}$$

We now need to understand what is meant by scale invariance. Let us define the average gravitational potential in a ball of radius $R$ to be

$$\Phi(R) = \frac{1}{V_R}\int_{V_R} d^3x\Phi(\vec{x})$$

where $V_R$ is the volume of the ball. We can define the variance of $\Phi$ on that scale to simply be

$$\sigma^2_R(\Phi) = \langle\Phi^2(R)\rangle$$

where $\langle\cdots\rangle$ is an ensemble average, i.e. an average over many possible configurations of $\Phi$. A scale invariant spectrum corresponds to a variance which is independent of $R$, i.e. $\sigma^2_R(\Phi) \propto$ constant. It turns out that we can relate $\sigma^2_R(\Phi)$, to $\delta(t,\vec{k})$ through the Newton-Poisson equation. Indeed we have that

$$\sigma^2_R(\Phi) \simeq \frac{k^3}{4\pi}\langle|\Phi(t,\vec{k})|^2\rangle \propto \frac{k^3}{k^4}\langle|\delta(t,\vec{k})|^2\rangle \quad \text{with} \quad k = \frac{2\pi}{R} \tag{11}$$

If it is scale invariant we then have that the power spectrum of the density fluctuations at initial time $t_i$, $P_i(\vec{k}) \equiv \langle|\delta(t_i,\vec{k})|^2\rangle$ has the form

$$P_i(k) \propto k$$

In practice, choosing scale invariant initial conditions for the density field corresponds to picking the amplitude of the density field[5] to be given by $|\delta(t,\vec{k})| \propto k^{1/2}$.

[4]This is only strictly true of the random process is Gaussian. For non-Gaussian processes one has to go further and characterize such quantities as $\langle\delta(\vec{x}_1)\delta(\vec{x}_1)\delta(\vec{x}_1)\rangle$ and higher order products. It turns out the Inflation predicts that the random fields are, to a very good approximation, Gaussian.

[5]This is not strictly true, otherwise we would have $\langle\delta\rangle \neq 0$ but given that we are not interested in $\delta_{\vec{k}}$ today, but in $P(k)$, this prescription will suit us

Having chosen a set of initial conditions we can predict what the large scale structure of the Universe for different sets of assumptions. We will do so for three model universes: a universe where matter is solely of baryons and known as the Baryonic Dark Matter (BDM) universe ; a universe which also contains pressureless dark matter and is known as the Cold Dark Matter (CDM) universe; and a universe in which matter is dominated by massive neutrinos and is known as the Hot Dark Matter (HDM) universe.

## 4.1 A Baryonic Dark Matter Universe

Consider the simplest scenario we can imagine with the tools we have been give: a flat Universe consisting solely of photons, atoms (i.e. Baryons) and a cosmological constant. There is a limit on how much of the Universe can be made of baryons: the abundance of light elements restricts $\Omega_b h^2 \simeq 0.024$. With our current constraints on the Hubble constant, this means that fractional energy density in baryons must be around 5% and given that we are considering a Universe with $\Omega = 1$ we must have $\Omega_\Lambda = 0.95$. In figure 1 we plot the power spectrum of such a theory and we can clearly identify the main features. On very large scales (i.e. on scales larger than the sound horizon at equality between matter and radiation) perturbations will grow until they reach the $\Lambda$ era, after which they will be constant. On scales below the sound horizon, perturbations will initially grow, then oscillate acoustically and once the Universe recombines, they will grow again until the freeze in during the $\Lambda$ era. Hence we see a series of peaks and troughs on intermediate to small scales. On very small scales, i.e. on scales which are smaller than the Silk damping scale at recombination, perturbations are severely supressed and we can see exponential damping.

Putting all the pieces together, we find that the mass variance for this theory is roughly described by

$$\begin{aligned} \Delta^2(k) &\propto k^4 \quad \text{if } k\eta_* < 1 \\ \Delta^2(k) &\propto \frac{\cos^2(\frac{k\eta_*}{\sqrt{3}})}{k} e^{[-2(k/k_S)^2]} \quad \text{if } k\eta_* > 1 \end{aligned} \tag{12}$$

We can also consider a Universe in which, for some reason, we have underestimated the density of baryons. If we assume that the majority of baryons are dark (in the form of dark nuggets of matters, brown dwarfs or even black holes) we can choose $\Omega_B = 0.25$ and $\Omega_\Lambda = 0.75$. In Figure 1 we plot the power spectrum of this theory which we can call the Baryonic Dark Matter scenario (or BDM for short) and we can still the gross qualitative features we identified above.

## 4.2 A Cold Dark Matter Universe

An interesting scenario arises if we consider a Universe in which, once again, we have radiation, baryons and $\Lambda$ but now add a component of pressureless, non-relativistic matter that does not interact with the radiation. We shall call this the Cold Dark Matter scenario (or CDM) and has a qualitatively different behaviour to the BDM case. There are effectively two regimes that will define the shape of the power spectrum. On very large scales, i.e. scales such that $k\eta \ll 1$

for all times before equality between radiation and matter, the density contrast will grow as $\eta^2$ until it reaches the $\Lambda$ dominated era. For scales that cross this threshold, i.e. such that $k\eta_{eq} < 1$, the density contrast will have its evolution surpressed to logarithmic growth; this surpression in growth will last between the time the wavelength of the perturbation has the same size as the sound horizon and the transition from radiation to matter domination. During the matter era, perturbations will grow again until $\Lambda$ domination sets in.

A rough estimate of the mass variance, $\Delta^2(k)$ gives us

$$\begin{aligned} \Delta^2(k) &\propto k^4 \quad \text{if } k\eta_{eq} < 1 \\ \Delta^2(k) &\propto (\ln(k))^2 \quad \text{if } k\eta_{eq} > 1 \end{aligned} \tag{13}$$

The overall shape can be clearly seen in Figure 1.

### 4.3 A Hot Dark Matter Universe

There is yet another simple model we can consider. If we now replace the pressureless matter in the CDM model by light massive neutrinos, we will have an altogether different cosmology. The motivation is clear: we know that neutrinos exist and there is even tentative evidence that they may have a mass. As we saw in the previous section, neutrinos will not evolve as a fluid and will free stream while they are relativistic, exponentially damping all perturbations on small scales. The neutrinos are weakly interacting, dark (i.e. they don't interact strongly with light) and move relativistically so can be considered a "hot" component of the Universe. For these reasons, a Universe in which neutrinos make up the bulk of the energy density today is called the Hot Dark Matter scenario (or HDM). The mass variance can be roughly approximated to

$$\Delta^2(k) \propto k^4 \exp(-k\lambda_{FS}) \tag{14}$$

and is plotted in Figure 1.

## 5 Comparing to observations

Furthermore, we have been talking about $\delta$, a continuous field defined over all of space. But what we actually see are galaxies, bright dots in the sky. We must relate our theory, the theory of how $\delta$ arises and evolves with our data, a catologue of positions of galaxies in the sky. Once again let us take as our starting point, the mean density, $\rho_0$ and expand it in terms of the density contrast:

$$\rho = \rho_0(1 + \frac{\delta\rho}{\rho}) = \rho_0(1 + \delta)$$

The galaxies must trace this density somehow. For example, if the density is high (i.e. where there is an overdensity) we expect to see more galaxies, more bright dots. If the density is low, we expect to see less galaxies, a void. We can make this comparison more quantitative.Take the distribution of galaxies in the Universe and lay down balls of radius $\lambda$ all over. Calculate the mass contained in each ball (i.e. add up the mass in all the galaxies contained in each ball). We have that

$$M(\lambda) = \int_{S_\lambda} d^3x\rho(\vec{x})$$

where $S_\lambda$ is a sphere of radius $\lambda$. We can find the average over all balls to get

$$\bar{M}(\lambda) \equiv \langle M(\lambda)\rangle = \frac{4\pi}{3}\lambda^3\rho_0$$

Suppose we now calculate the variance

$$[\delta M(\lambda)]^2 \equiv \langle [M(\lambda) - \bar{M}(\lambda)]^2\rangle$$

Some balls will be heavier and others will be lighter. There should be a scatter. One can show that the mass variance is given by

$$\Delta^2(k) \equiv \left[\frac{\delta M(\lambda)}{\bar{M}(\lambda)}\right]^2 \simeq \frac{k^3}{2\pi^2}P(k) \quad \text{with } k = \tfrac{2\pi}{\lambda}$$

So $P(k)$ gives you the fluctuations in mass of balls with a given radius. Different types of clumping will lead to different $P(k)$ and these can be compared to the clumping that we see in the distribution of galaxies.

A typical survey of galaxies will be like the image in Figure 2 If we were to consider a universe with baryons and radiation, as described above, we would find a power spectrum as in Figure 1.

Clearly the problem is that perturbations in the baryons are prevented from growing because of the way they link up to radiation. One solution is to have a different form of non-relativistic matter that doesn't interact with radiation. It won't feel that baryon pressure and will have its Jean's length equal to zero. Because this matter does not interact with light, it can be called "dark matter". An example of the mass variance $\Delta(k)^2$ compared to data is shown in Figure 3.

## Anisotropies in the Cosmic Microwave Background

We have focused on the evolution of perturbations in the density field and how we can connect them to the distribution of galaxies. There is, of course, another very important component to the Universe, which plays a significant role at early times: radiation.

Let us briefly recall the history of radiation in the Universe. At very early times the universe is highly ionized, Hydrogen is dissociated into free electrons and protons and the mean free path of photons is effectively 0. The universe is opaque. At late times, the universe is neutral, protons and electrons are in Hydrogen atoms and photons are free to propagate. The Universe is transparent. The transition from one state to another is naturally related to the binding energy of hydrogen and the evolution of the ionization fraction

$$X \equiv \frac{n_p}{n_p + n_H} \tag{15}$$

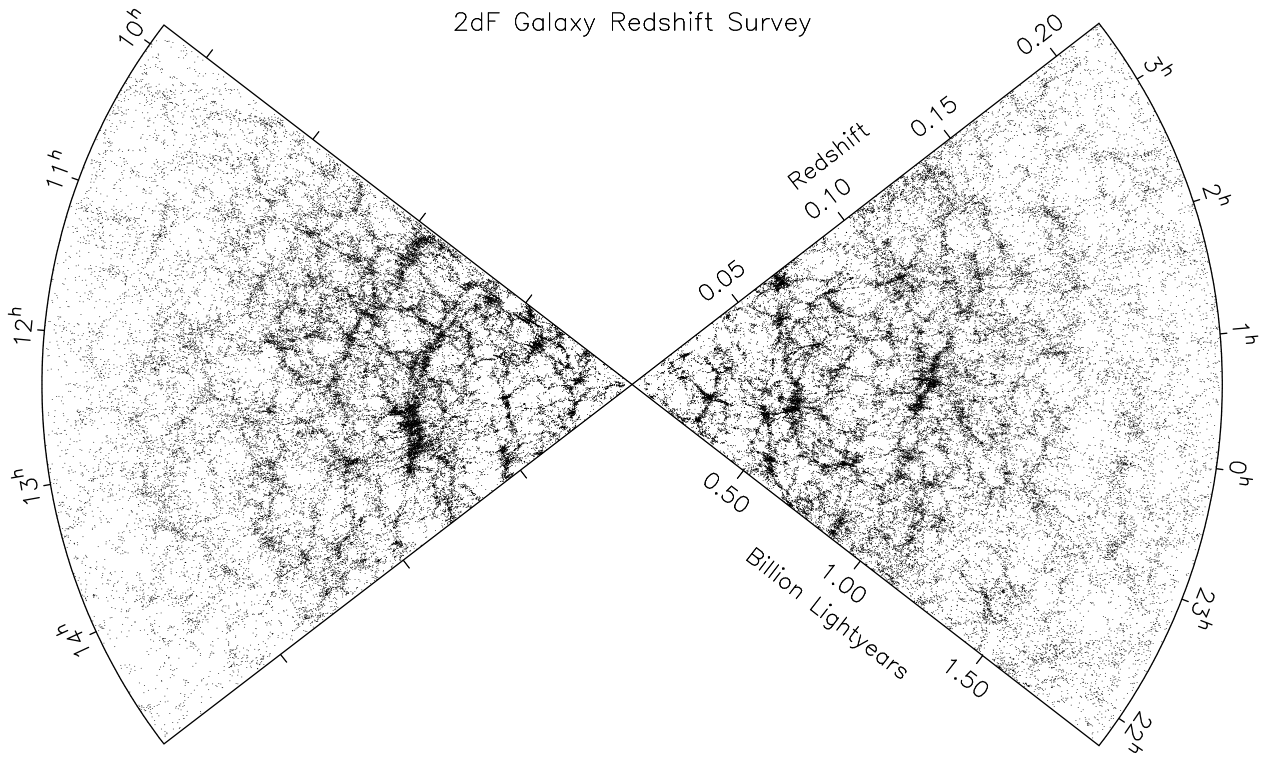


Figure 2: A survey of galaxies over the sky illustrates there are inhomogeneities spanning a wide range of scales (2dF Galaxy Redshift Survey -based at Oxford)

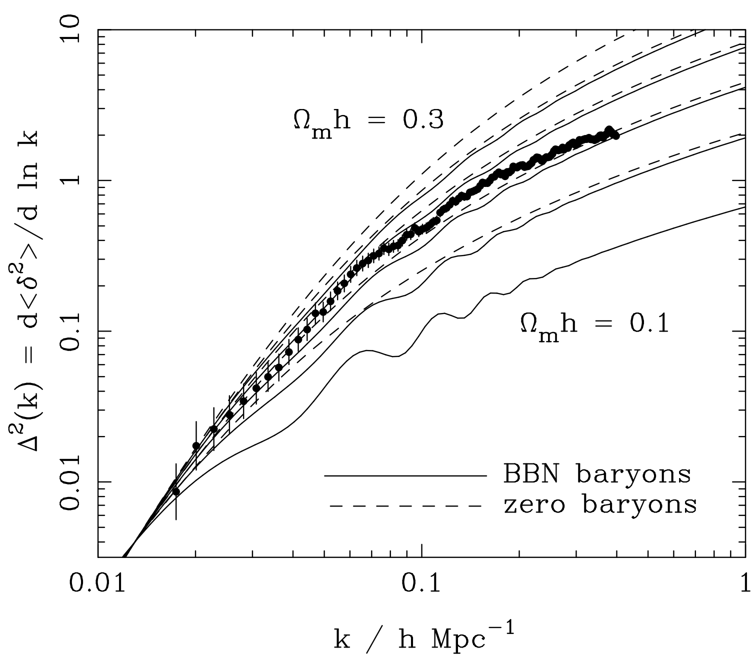


Figure 3: The mass variance, $\Delta^2(k)$ of inhomogeneities with cold dark matter compared to the real universe

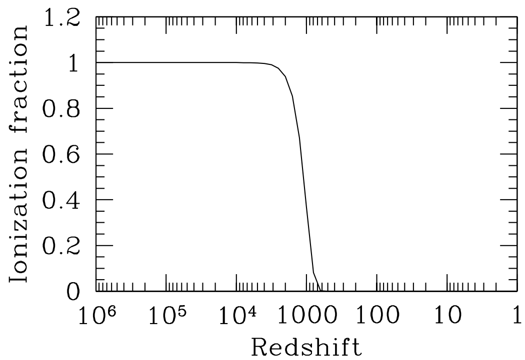


Figure 4: The evolution of the ionization fraction as a function of redshift

At sufficiently early times we will find that $X = 1$, i.e. the Universe is completely ionized. As it crosses a certain threshold, electrons and protons combine to form Hydrogen. This happens when the temperature of the Universe is $T \simeq 3570K$ or $0.308eV$, i.e. when it was approximately $380,000$ years old, at a redshift of $z \simeq 1100$. We would naively expect this to happen at $13.6eV$. One way to think about why this isn't so is that, at a given temperature there will always be a few photons with energies larger than the average temperature. This energetic photons only become unimportant at sufficiently low temperatures.

We can now reconstruct the history of a photon left over from recombination. We know that, post recombination, a photon has been travelling along a straight path from then until now. It has travelled a distance $d_* \simeq \eta_0 - \eta_*$ where $\eta_0$ $(\eta_*)$ is the conformal time today (at recombination). Before recombination, the mean free path of the photon was negligable, it was effectively standing still. So we can think of recombination as the time when these photons were released to travel through space. When we look from a fixed point in the sky, we will received photons that have travelled straight towards us since $\eta_*$, the point at which they are released will map out a two dimensional sphere, of radius $d_*$ which is known as the surface of last scatter. We can think of this light as an image of the the surface of last scatter, a photograph of a spherical slice of the universe at $\eta_*$.

Recall that the Universe was in thermal equilibrium and hence the radiation should have a black body spectrum. And given that it is approximately homogeneous on large scales, the black body will be the same in all directions, with the peak at the same temperature in all directions. But we also know that it perturbed and hence we should see deviations. These deviations will have various contributions.

For a start, we have the Stefan-Boltzman law: $\rho_\gamma = \sigma T^4$ where $\sigma$ is the Stefan Boltzman constant. But we then have

$$\frac{\delta T}{T} = \frac{1}{4}\frac{\delta\rho_\gamma}{\rho_\gamma}.$$

This seems quite easy to understand: the more compressed the radiation is, the hotter it is. So points on the surface of last scattering which have denser radiation will look hotter. This is known as the *intrinsic term*

When the photons are released at the surface of last scattering, the will collide with one last electron or proton before they propagate towards us. That proton or electron may have a peculiar velocity, $\vec{v}_B$. This will impart a Doppler shift on the photon, i.e. the observed temperature will be $T' = T(1 - \vec{v}_B \cdot \vec{n})$ so that

$$\frac{\delta T}{T} = -\vec{v}_B \cdot \vec{n}$$

This is known as the *Doppler* term.

There will be gravitational effects too. If the photon is caught in a gravitational well at the surface of last scatter, it will be held back, i.e. gravitationally redshifted. The bigger the well, the colder the photon becomes:

$$\frac{\delta T}{T} = -\Phi$$

This is known as the *Sachs-Wolfe* term. Finally, as the photon propagates towards us trhough empty space, space time is changing and warping as it evolves. The photon will be redshifted or blue shiftet according to:

$$\frac{\delta T}{T} = -2\int_{\eta_*}^{\eta_0} d\eta \dot{\Phi} \tag{16}$$

This is known as the *Integrated Sachs-Wolfe* term.

Let us put all this together now. Recall: the surface of last scatter is a sphere with radius $d_*$. Suppose we look in a give direction $\hat{n}$. We will see

$$\begin{aligned}\frac{\delta T}{T}(\hat{n}) &= \frac{1}{4}\delta(\eta_*, d_*\hat{n}) - [\vec{v}_B \cdot \hat{n}](\eta_*, d_*\hat{n}) - \Phi(\eta_*, d_*\hat{n}) \\ &\quad -2\int_{\eta_*}^{\eta_0} d\eta' \dot{\Phi}[\eta', (\eta_0 - \eta')\hat{n}]\end{aligned} \tag{17}$$

In summary, a measurement of $\frac{\delta T}{T}$ is a snapshot of the universe at $t_*$, it is related to quantities that we know from studying large scale structure and we can use it to do "archeology"of the Universe.

What do we expect to see? First of all, recall the we looked at how perturbations in the radiation and baryons evolve in the radiation era, before recombination. On small scales, there should be a series of acoustic oscillations. We found a solution, $\delta_\gamma \propto J_2(k\eta/\sqrt{3})$. We also know that $\vec{\nabla} \cdot \vec{v} = -\dot{\delta}$ So if $\delta_\gamma \propto \cos(k\eta/\sqrt{3})$ then $\vec{v}_B \cdot \hat{n} \propto \sin(k\eta/\sqrt{3})$. We should see an oscillatory pattern in $\frac{\delta T}{T}$ with the spacing between peaks and troughs set by the angular projection of $k\eta_*/\sqrt{3}$. On very large scales we expect it to be relatively featureless.

On small scales, an altogether different phenomenon kicks in. During recombination, as the photons decouple from the baryons, they will slowly start to propagate. They will move around a little and in doing so, they will leak energy from high density regions to low density regions. The net effect is to reduce high density regions, fill in low density regions and effectively smooth out perturbations. features on small scales will be smoothed out.

How do we analyse a map of the cosmic microwave background? We need to take the equivalent of the fourier transform except now it is on the surface of the sphere. You will recall from mathematical methods and quatum mechanics that there is a useful basis to this in, the Spherical Harmonic functions. So we can take

$$\frac{\delta T}{T}(\hat{n}) = \Sigma_{\ell m} a_{\ell m} Y_{\ell m}(\hat{n}). \tag{18}$$

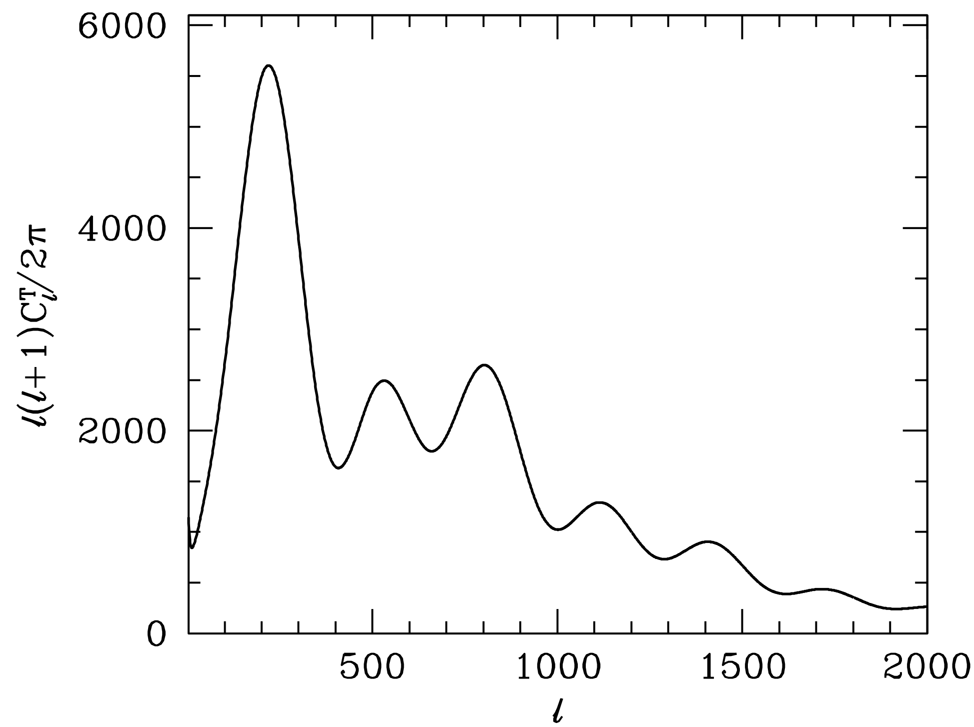


Figure 5: The powerspectrum of the CMB

and plot the powerspectrum

$$C_\ell = \frac{1}{2\ell+1}\Sigma_m |a_{\ell m}|^2 \tag{19}$$

In Figure 5 you can see the structure of the power spectrum. There is clear evidence of the oscillatory structure as well as of the damping on very small scales. In Figure 6 we can see how remarkably well the power spectrum has been measured.

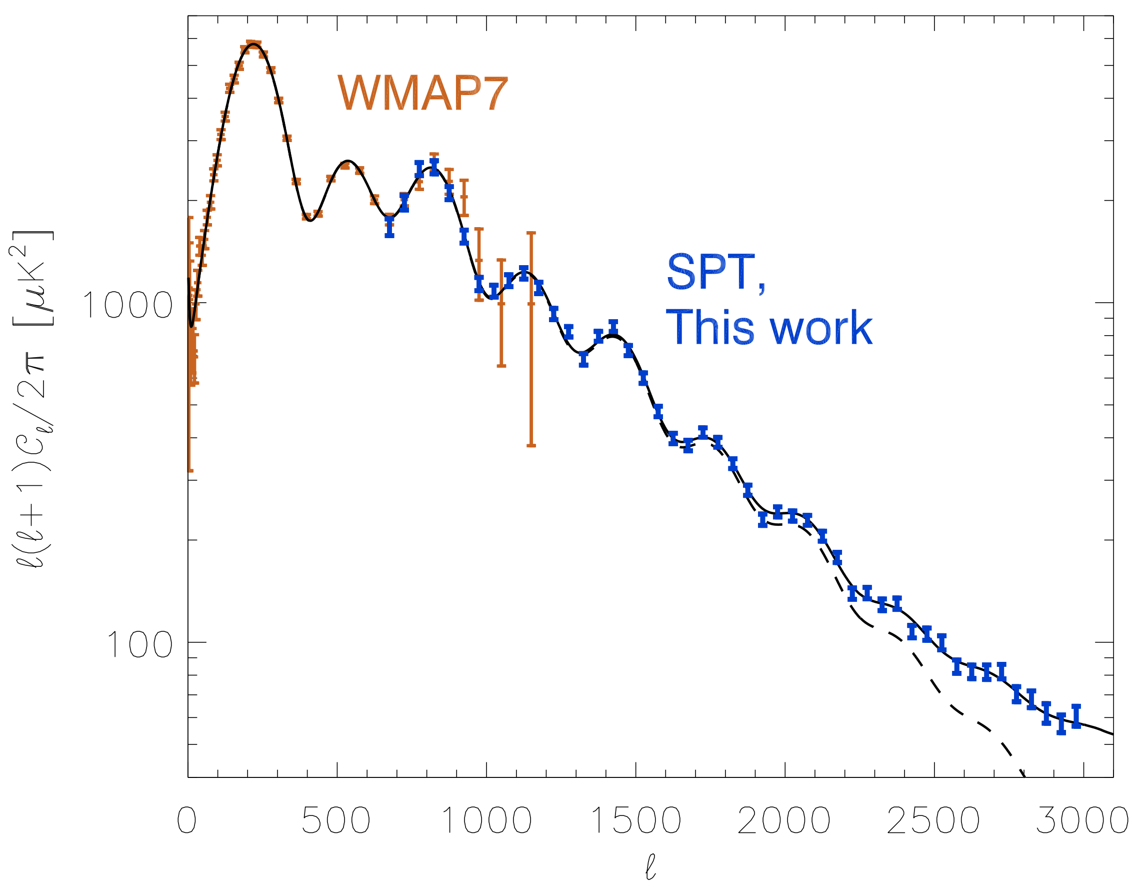


Figure 6: The best fit powerspectrum of the CMB as compared to the WMAP and SPT data (Keisler 2011)