1. (a) Explain what is meant by the *luminosity* and the *effective temperature* of a star. Discuss briefly the importance of these quantities in studies of stellar structure and evolution.

[5]

(b) A model star of mass M and radius R consists of a fully convective interior surrounded by a thin, low density photosphere in radiative equilibrium. The temperature of the photosphere may be taken as constant and equal to the effective temperature, T_{eff}. If the transition between photosphere and interior occurs at optical depth $\tau = 2/3$ and the opacity κ of the stellar material varies with pressure P and temperature T as $\kappa = \kappa_0 P T^5$, where κ_0 is a constant, show that the pressure P_{ph} at the base of the photosphere is given by

$$T_{eff}^5 P_{ph}^2 = 4GM/(3\kappa_0 R^2).$$

[10]

(c) Obtain a further relation between P_{ph} and T_{eff} by using the condition that the total convective flux F_C must equal the total radiative flux F_R at the base of the photosphere. You may assume that the convective flux is given by $F_C = 73.3\mu^{-1/2} P T^{1/2}$, where μ is the mean relative molecular mass of the stellar material.

[5]

(d) Hence show that for model stars of solar composition ($\mu = 2/3; \kappa = 6.9 \times 10^{-27}$ m^2 kg^{-1}) the luminosity L is related to the mass and effective temperature by

$$\log(L/L_\odot) = -8\log(T_{eff}/10200) + \log(M/M_\odot).$$

At what stage in the life of a star with mass $M = M_\odot$ might such an evolutionary track be appropriate? [$\sigma = 5.67 \times 10^{-8}$ Wm^{-2} K^{-4}; $L_\odot = 3.85 \times 10^{26}$ W; $M_\odot = 1.99 \times 10^{30}$ kg]

[10]

2. (a) Consider the accretion of matter onto a black hole of mass M, producing an accretion luminosity L. Assuming spherical accretion, derive the maximum luminosity, known as the Eddington limit $L_{\rm Edd}$, above which no more matter can be accreted:

$$L_{\rm Edd} = \frac{4\pi GMc}{\kappa},$$

where κ is the opacity of the accreted material.

[8]

(b) In the case of a black-hole binary or in the case of accretion onto the central black hole in a quasar, accretion is believed to take place through an accretion disc where the inner radius of the disc is given by $R_{\rm in} = (M/M_\odot) \times 9\,{\rm km}$.

Now suppose that roughly half of the accretion luminosity is released within a radius of $2\,R_{\rm in}$, ignore temperature variations across this region of the disc and assume that the black hole accretes at the Eddington limit. Estimate the temperature in the inner part of the accretion disc, both for a stellar-mass black hole of mass $M = 10\,M_\odot$ and for a supermassive black hole of mass $M = 10^8\,M_\odot$, taking $\kappa = 0.034\ m^2\ kg^{-1}$.

[6]

(c) In what part of the electromagnetic spectrum is it best to observe the inner disc in the case of (a) a stellar-mass black hole and (b) a supermassive black hole?

[2]

(d) Suppose that a quasar containing a supermassive black hole whose mass is $10^9\,M_\odot$ (as inferred from the widths of its broad emission lines) is found at redshift $z = 10$. Discuss whether this presents a challenge to the notion that the black hole's mass has been built up via Eddington-limited accretion.

[4]

3. Assume for simplicity that $a(t) \propto t$. Recall that the energy of a particle is $E = mc^2$, and of a photon $E = 3kT$. The photon-to-baryon ratio in the Universe is 10^9.

   (a) The temperature at the core of the Sun is around 10^7 K. How old was the Universe when it was this hot? Was it matter-dominated or radiation-dominated at that time (= which component had higher energy density)?

   [10]

   (b) What was the typical energy of free electrons? Were these relativistic or not?

   [5]

   (c) At the CERN collider, typical particle energies are of order of 100 GeV. How old was the Universe when typical particle energies were around this size? What was the temperature at this time?

   [5]

4. (a) The Galaxy's age can be estimated by radioactive decay of Uranium. Uranium is produced as an r-process element in supernovae, and on this basis the initial abundances of the two isotopes U^{235} and U^{238} are expected to be in the ratio

$$\left(\frac{U^{235}}{U^{238}}\right)_{\text{initial}} \approx 1.65 \ .$$

The decay rates of the isotopes are

$$\lambda(U^{235}) = 0.97 \times 10^{-9}\ \text{yr}^{-1}$$
$$\lambda(U^{238}) = 0.15 \times 10^{-9}\ \text{yr}^{-1}$$

The present-day abundance ratio is $U^{235}/U^{238}_{\text{final}} \approx 0.0072$. Use the decay law

$$U(t) = U(0)\exp(-\lambda t) \ ,$$

to estimate the age of the galaxy, assuming the galaxy took an additional 10^9 yrs to form before the peak of the supernova activity. Obtain an upper limit on the value of the Hubble parameter h assuming a critical-density Universe.

[10]

(b) In a dust-dominated open Universe, the present age of the Universe is given by the appalling formula

$$H_0 t_0 = \frac{1}{1-\Omega_0} - \frac{\Omega_0}{2(1-\Omega_0)^{3/2}} \cosh^{-1}\left(\frac{2-\Omega_0}{\Omega_0}\right) \ .$$

Demonstrate that in the limiting case of an empty Universe $\Omega_0 \to 0$ we get $H_0 t_0 = 1$, and in the limiting case of a flat Universe $\Omega_0 \to 1$ we recover the result $H_0 t_0 = 2/3$.

[Useful formulae: $\cosh^{-1}(x) \approx \ln(2x)$ for large x, and $\cosh^{-1}[(1+x)/(1-x)] \approx 2\sqrt{x} + 2x^{3/2}/3$ for small x]

[10]

(c) Give a physical argument explaining why introducing a positive cosmological constant will increase the predicted age of the Universe.

[5]