SECOND PUBLIC EXAMINATION

Honour School of Physics – Part B: 3 and 4 Year Courses

Honour School of Physics and Philosophy Part B

B3: Astrophysics & Atmospheric physics

Answer **five** *questions with at least* **two** *from each section:*

Start the answer to each question on a fresh page.

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight which the Examiners expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section A (Astrophysics)

1. Using the condition of hydrostatic equilibrium show that for a star whose equation of state is given by the ideal gas law, its central temperature (T_c) can be estimated as

$$kT_c \simeq \frac{GM\,\mu m_H}{R},$$

where M is the mass of the star, R its radius and μ the mean molecular weight of the stellar material [$\mu \simeq 0{\cdot}6$ for a star like the Sun]. Estimate T_c for the Sun and comment on what the result implies for the energy production in the centre of the Sun. [7]

Use this result and the equation of radiative transfer to show that the mass–luminosity relation for low- and intermediate-mass stars can be written as

$$L \propto \frac{\mu^4 M^3}{\langle\kappa\rangle},$$

where L is the star's luminosity and $\langle\kappa\rangle$ is a characteristic average opacity of the stellar material. [5]

Consider a fluid element at the surface of a star. Show that this element experiences an outward force due to the momentum deposited in it by radiation. By balancing this force with the gravitational force (or otherwise), show that there is a maximum luminosity for a star of mass M, known as the Eddington limit,

$$L_{\rm Edd} = \frac{4\pi GMc}{\kappa}.$$ [6]

Discuss what happens when a star as a result of its internal evolution reaches this maximum luminosity? [2]

2. Starting from the equation of hydrostatic equilibrium, show that the relation between the total thermal energy (U) and the total gravitational energy (Ω) of a star is given by

$$3(\gamma - 1)\,U + \Omega = 0$$

where γ is a mean adiabatic exponent of the stellar material (i.e. the ratio of specific heats for an ideal gas). [8]

For a fully, ionized, ideal gas, $\gamma = 5/3$. Explain why this implies that stars effectively have a negative heat capacity. [4]

Use this fact to explain

(a) why nuclear burning in stars like the Sun is stable (i.e. does not lead to a nuclear runaway);

(b) the basic principles governing the evolution of stars in phases when nuclear burning is not important (from the pre-main-sequence phase to the supernova stage). [8]

3. What is meant by the term *decoupling*? Describe how the Universe was different before and after decoupling. [5]

Given that there are $\sim 10^9$ photons per baryon, estimate the temperature at which decoupling took place. Are the electrons relativistic during decoupling? [10]

Assuming that the relationship between the temperature T and cosmic time t is given by $T \propto t^{-2/3}$, estimate the age of the Universe at decoupling. [5]

4. What is the currently accepted range of values for the density of the Universe at the present day in terms of the critical density? Indicate where these limits come from. [8]

The relationship between temperature T and scale factor $a(t)$ is given by $T \propto 1/a(t)$. Describe how $a(t)$ behaves if the Universe is (a) matter dominated and (b) radiation dominated, in the absence of curvature. The temperature at matter–radiation equality is 16 500 K and the age is 10^{11} s. At what age was the temperature 3×10^{25} K?

Suppose that at that time the density parameter Ω was somewhat less than 1, so that the Universe quickly became curvature dominated, with expansion law $a(t) \propto t$. How would the relation between temperature and time be changed? How old would the Universe have been when its temperature fell to 3 K? Comment on the answer you obtain. [12]

Section B (Atmospheric Physics)

The Navier-Stokes equation for a viscous fluid is

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}.\nabla)\mathbf{u} + \frac{1}{\rho}\nabla p = \nu \nabla^2 \mathbf{u},$$

where $\mathbf{u}$ is the fluid velocity, ρ the density, p the pressure and ν the kinematic viscosity.

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5. Show that the effective molar mass M of a mixture of gases with molar masses M_1 and M_2 and partial pressures p_1 and p_2 is given by

$$M = \frac{M_1 p_1 + M_2 p_2}{p}$$

where p is the total pressure. Calculate the effective molar mass of air given that it is composed of nitrogen, oxygen and argon in proportions 78:21:1 parts by volume and that the molar masses of the constituents are 28, 32 and 40 g respectively. [6]

Assuming that the atmosphere is in hydrostatic equilibrium and behaves as a perfect gas, show that the pressure p at a height h above the surface is given by

$$p = p_0 \exp\left(- \int_0^h \frac{Mg}{RT}\, \mathrm{d}z \right)$$

where p_0 is the surface pressure, R is the molar gas constant and T is the temperature at height z. [8]

Given that temperature falls with height at a uniform rate of 7 K km^{-1} from a surface temperature of 290 K and that the air contains 10% by volume of water vapour, calculate the height (in km) at which the pressure has fallen to 80% of its surface value. [6]

[Molar mass of water vapour = 18 g.]

6. What is meant by the *adiabatic lapse rate* in an atmosphere? Why is it important in considerations of hydrostatic stability? [5]

Show that in a dry atmosphere, the adiabatic lapse rate Γ_D is given by

$$\Gamma_D = \frac{Mg}{C_p}$$

where M is the relative molecular mass of the air, C_p is its molar heat capacity at constant pressure and g is the acceleration due to gravity. Does the presence of water vapour affect the lapse rate if the air is *not* saturated? [6]

In the case of a saturated atmosphere, by consideration of the first law of thermodynamics and the Clausius-Clapeyron equation or otherwise, show that the adiabatic lapse rate Γ_S is given by

$$\Gamma_S = \Gamma_D \frac{1 + Le/RTp}{1 + L^2e/RC_pT^2p}$$

where T is temperature, p is total pressure, e is the saturation vapour pressure of water, L is its molar latent heat and R is the molar gas constant. Assume that $e \ll p$ and that the volume of liquid water can be neglected. [9]

7. Derive the expression

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}$$

for the acceleration of a moving fluid element, where $\mathbf{u}(\mathbf{r}, t)$ is the fluid velocity at a fixed point $\mathbf{r}$ at time t. Comment briefly on the implications of the term $(\mathbf{u} \cdot \nabla)\mathbf{u}$ for the time-development of fluid flows. [7]

A viscous fluid of uniform density ρ flows steadily in the x-direction with speed $u(z)$ and is subject to a constant pressure gradient dp/dx in the x-direction. It is bounded below and above by wide, fixed plates at $z = 0$ and $z = d$, respectively. (This is an example of *Poiseuille flow.*) Find an expression for the flow profile $u(z)$ in terms of the pressure gradient and the dynamic viscosity, and sketch $u(z)$. [8]

Give an expression for the Reynolds number for this flow. Explain briefly what is observed experimentally for flow in a pipe as the imposed pressure gradient is increased from zero. [5]

8. By using a scale analysis, show that slow, viscous flow under certain conditions (which you should state) can be represented by the equation for *Stokes flow*,

$$\nabla^2 \boldsymbol{\omega} = 0 \ ,$$

where $\boldsymbol{\omega}$ is the vector vorticity field. [6]

Using dimensional analysis, show that the drag force D on a solid body of typical dimension a, moving slowly at speed U though a fluid of dynamic viscosity μ that is at rest far from the body is given by

$$D = K \mu a U \ ,$$

where K is a dimensionless constant that depends on the shape of the body.

How would you expect this drag formula to be modified if the solid body is replaced by a drop of a different fluid, of dynamic viscosity μ'? [7]

When the solid body is a sphere, $K = 6\pi$. Calculate the terminal velocity of a spherical raindrop of radius 10 μm falling in air. Verify that the assumption of Stokes flow is in fact valid in this case. [Assume that the density of air is $1{\cdot}3\,\mathrm{kg\,m^{-3}}$ and that its dynamic viscosity is $1{\cdot}7 \times 10^{-5}\,\mathrm{kg\,m^{-1}\,s^{-1}}$.] [7]