Early Universe and Galaxy Problems

Set by Dr. Pedro G. Ferreira for the 4^{th} year Astrophysics course.

Problem 1: Units

Convert the following quantites by inserting the appropriate factors of c, $\hbar$, k_B and unit conversions.

- H_0 =100$h$km s^{-1}Mpc^{-1} into (a) eV, (b) Mpc^{-1}, (c) Gyr^{-1}. These correspond to an upper limit on the mass of the dark energy particle, the inverse Hubble length, inverse approximate age.
- $\rho_{crit} = 3H_0^2/8\pi G$ into (a) g cm^{-1}, (b) GeV^4, (c) eV cm^{-3}, (d) protons cm^{-3}, (e) M_{Sun}Mpc^{-3}. If the cosmological constant has $\rho_\Lambda = 2\rho_{crit}/3$, what is its energy scale in eV (i.e. $\rho_\Lambda^{1/4}$). Compare to the Planck mass, $M_{Pl} = (8\pi G)^{-1/2}$
- The photon temperature, T_{CMB}=2.728K to (a) eV. Assuming a black body distribution, convert this to a number density, n_γ in photons cm^{-3} and energy density, ρ_γ in (a) eV, (b) g cm^{-3} and $\Omega_\gamma = \rho_\gamma/\rho_{crit}$.
- The neutrino temperature, $T_\nu = (4/11)^{1/3}T_{CMB}$. Use this to express n_ν, ρ_ν and Ω_ν in the above units assuming that the neutrinos are relativistic (and have three species).
- With the above relic *number* density, now consider the case where one out of three neutrino species has a mass of 1 eV and the rest are massless. What is the energy density of relic neutrinos in units of the critical density, $\Omega_{\nu,massive}$. For what mass is the energy density at the critical value?

Problem 2: Age

- Assume the universe today with both matter (Ω_m) and a cosmological constant (Ω_Λ). Compute the age of the universe and plot your result for t_0 in h^{-1} Gyr as a function of Ω_m.
- Assume instead that the universe is open with matter (Ω_m) but no cosmological constant. Compute the age of the universe and plot your result for t_0 in h^{-1} Gyr as a function of Ω_m.
- Assume that there is only matter and radiation in the universe (no cosmological constant) and that the universe is flat. Integrate the age equation to determine the time at which the cosmic temperature was 10^9K and 1/3 eV.

Problem 3: Conformal Time

- Assume the universe today is flat with both matter (Ω_m) and a cosmological constant (Ω_Λ). (a) Compute the conformal age or horizon of the universe and plot your results for $H_0\eta_0$ as a function of Ω_m. (b) What is the current horizon size for a universe with $\Omega_m = 1/3$ and $h = 1/\sqrt{2}$? (c) What is the mass contained within the current horizon in solar masses? If all objects were $10^{13}h^{-1}M_{sun}$ in mass, how many are in the observable universe?

Problem 4: Comoving Distance and Horizon

- Write down the expression for the conformal time elapsed between some initial epoch $a_i = (1+z_i)^{-1}$ and a final epoch, $a_f = (1+z_f)^{-1}$. This is also the distance a particle going at the speed of light travels in this interval in comoving coordinates. At what redshift has light travelled halfway across the current horizon?

- In a flat, $\Omega_m = 1$ universe with no radiation, calculate the horizon scale $z = 1000$. What is the angular scale subtented by this scale today? Express your result in degrees and angular frequency $\ell \equiv 2\pi/\theta$ (where θ is in radians). The CMB is smooth above this scale and this is known as the horizon problem.

High-Energy Astrophysics Problems

by Dr Garret Cotter for the 4^{th} year Astrophysics course.

Problem 1: Eddington luminosity

For each of the following objects, estimate the ratio of the object's luminosity to its Eddington luminosity. Comment in each case.

- An adult human at rest.
- A 100-W light bulb.
- The sun.
- A quasar with a bolometric luminosity of 10^{40} W powered by accretion onto a $10^9\ M_\odot$ black hole
- Supernova 1987A: a 20 $M_\odot$ star which exploded with a peak bolometric luminosity of about 10^{40}W.

Problem 2: Accretion discs

- Calculate the optimal efficiency of accretion onto a stationary black hole. Assume that material finally falls into the black hole from a last stable circular orbit at $R = 3R_\mathrm{S}$, and approximate the depth of the potential well at this point to be Newtonian.
- Show that if matter accretes at a rate $\dot{m}$ in a thin disc onto an object of mass M, the rate of release of gravitational binding energy at radius R is given by:

  $$\frac{dE}{dt}(r) = \frac{1}{2}\frac{GM\dot{m}}{r^2}$$

  Hence show that the temperature in such a disc at a radius r may be estimated to be

  $$T = \left(\frac{GM\dot{m}}{8\pi r^3 \sigma}\right)^{1/4},$$

  by equating the rate of release of binding energy in an annulus between r and $r + dr$ with the black-body power radiated from the annulus via the Stefan-Boltzman law.

  Use this result to estimate the temperature at the inner edge of such a disc for a $3M_\odot$ and a $10^9 M_\odot$ black hole, in each case assuming that the inner edge is at $3R_\mathrm{S}$. Comment.

- *For enthusiasts: non-examinable* Show that consideration of the angular momentum flux *outward* in the disc leads to a slightly higher estimate of

  $$T = \left(\frac{3GM\dot{m}}{8\pi r^3 \sigma}\right)^{1/4}$$

  as discussed in Longair, pp 147–151.

Problem 3: Shocks

- Using the ideal gas law and the strong shock jump conditions, show that the temperature of gas downstream of a shock is given by

$$T_d = \frac{3}{16}\frac{mv_u^2}{k_\mathrm{B}}$$

  where m is the mean particle mass in the gas and v_u^2 is the bulk speed of the gas upstream of the shock.

  In the lectures we considered the blast wave of SN1993J, which had an intital expansion speed of 20 000 km s^{-1}. Assuming the interstellar medium around the supernova to be hydrogen plasma, estimate the intial temperature behind the blast wave.

- Hydrogen plasma falls radially onto a white dwarf, passing through a shock very close to the surface. Show that the temperature behind the shock is given by

$$\frac{k_\mathrm{B}T}{m_\mathrm{e}c^2} = \frac{3}{32}\frac{m_\mathrm{e}}{m_\mathrm{p}}\frac{R_\mathrm{S}}{R_*}$$

  where R_* is the radius of the star. What is $k_\mathrm{B}T$ if $R_* = 6000$ km and $M = M_\odot$?