Early Universe and Galaxy Problems

Set by Dr. Pedro G. Ferreira for the 4^{th} year Astrophysics course.

Problem 9: A baryon dominated Universe

In the very early Universe, before recombination, baryons and photons interact very strongly to form a tightly coupled fluid. Let us assume that the transition between radiation and matter domination occurs at the same time as recombination. Before recombination, the evolution equation for the perturbations in the baryon/photon fluid is

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - \frac{1}{3a^2}\nabla^2\delta - \left(\frac{\dot{a}}{a}\right)^2\delta = 0 \tag{1}$$

with $\dot{a}/a = 1/2t$. After recombination the evolution equation is

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - \frac{3}{2}\left(\frac{\dot{a}}{a}\right)^2\delta = 0 \tag{2}$$

with $\dot{a}/a = 2/3t$.

- Rewrite these equations in Fourier space by replacing ∇ by $-i\vec{k}$.
- Solve both of these equations in the long wavelength limit.
- Before recombination, there is a scale (let us call it $k_J = 2\pi/\lambda_J$ where λ_J is the Jeans length) that separates the large k behaviour from the small k behaviour. What is it? How does it evolve with time?

A very rough approximation to the solution of the evolution equations before recombination on small wavelengths is

$$\delta = C_1 \cos\left(\frac{1}{\sqrt{3}}k\sqrt{t}\right) + C_2 \sin\left(\frac{1}{\sqrt{3}}k\sqrt{t}\right)$$

where C_1 and C_2 are constants.

- In the pre-recombination period, how does a perturbation that starts off with a $k < k_J$ evolve (answer this question qualitatively)? What type of behaviour will it have at sufficiently late times?
- Start off with a perturbation with an amplitude A on scales much larger than Jeans scale at some very early time. Follow its evolution until today. You should find two types of behaviour: modes whose wavenumber k are *smaller* than k_J at recombination and modes whose wavenumbers are *greater* than k_J. Find an approximate solution at recombination by matching the large wavelength solutions to short wavelength solutions at the time when $k_J = k$.
- Define the powerspectrum of perturbations to be

  $$P(k) \equiv |\delta(t_0, \vec{k})|^2$$

  Sketch what you expect the power spectrum of perturbations to be today if we assume that initially all modes had the same amplitude at some early time.

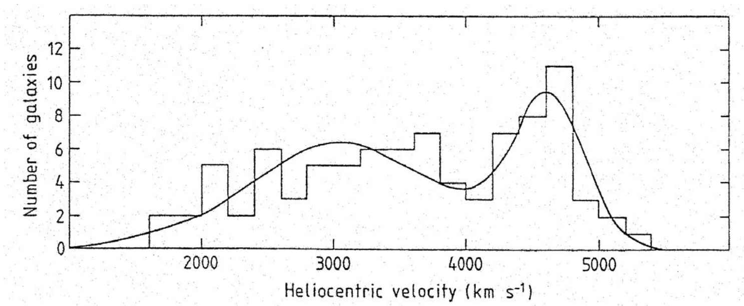

Figure 1: Recession velocities of galaxies (Problem 12)

Problem 10: Equation for a Spherical System

A spherical E galaxy has a density distribution:

$$\rho(r) = \frac{\rho_0}{1 + \frac{r^2}{a^2}}$$

Show that the enclosed mass $M(r) \propto r^3$ for $r \ll a$ and $M(r) \propto r$ for $r \gg a$. How does v_c depend on radius? What is the problem with this model for the density profile?

Now consider a simple model of the Galaxy in which matter is distributed in such a way that the rotation curve is flat with constant circular velocity v_c out to a cut-off radius r_* beyond which the density of matter is zero. Show that the escape speed v_e for stars with radii $r < r_*$ is given by

$$v_e^2(r) = 2v_c^2 \left[1 + \ln\left(\frac{r_*}{r}\right)\right]$$

The star G166-37 has the highest known radial velocity for a star in the solar neighbourhood at 430km/s. Using the model above, estimate a lower mass limit to the cut-off radius of the mass distribution and a lower limit to the total Galaxy mass.

Problem 11: The Galactic rotation curve

Let us assume we can split the density of the galaxy into three components. These are the bulge

$$\rho_b = \frac{A_b r^{\frac{3}{2}}}{(0.09 + r^2)}$$

the disk

$$\rho_d = A_d \exp\left(-\frac{r}{3.5}\right)$$

and the halo

$$\rho_h = \frac{A_h}{(21.2 + r^2)}$$

The mass in each of these components, out to radius of 50 kpc is $M_d = 10^{10} M_\odot$, $M_b = 2 \times 10^9 M_\odot$ and $M_h = 10^{11} M_\odot$. Sketch out the velocity rotation curve of the galaxy as function of radius. Do this approximately. At what distances does the halo begin to dominate?

Problem 12: The Virial Theorem

There is a dense grouping of galaxies in Centaurus filling a roughly circular area on the sky of radius 1^o. The figure shows a histogram of the radial velocities of the brighter galaxies, with the smooth line being the fit of two Gaussian functions to the histogram. The observations have been interpreted as showing two clusters at different distances superimposed along the line of sight. Assuming this is correct, obtain rough estimates for the distance between the clusters and the masses of the clusters. Assume that $H_0 = 50$ Km/s/Mpc.

High-energy astrophysics problems

Set by Dr. Garret Cotter for the 4^{th} year Astrophysics course.

Problem 6: Superluminal motion

In the lectures we derived the formula for the apparent projected velocity $\beta_{\rm app}c$ of material in a jet with velocity βc at an angle θ to the line of sight,

$$\beta_{\rm app} = \frac{\beta \sin\theta}{1 - \beta\cos\theta}$$

Use this formula to show that for any particular jet, the maximum apparent speed is

$$\beta_{\rm app}^{\rm max} = \gamma\beta,$$

which is seen when the jet is viewed from a direction such that

$$\beta = \cos\,\theta.$$

Problem 7: Relativistic Doppler beaming

This is a demonstration of the relativistic abberation of solid angle.

Suppose a source is moving along the $\theta = 0$ axis (spherical polar coordinates). If the source emits photons isotropically in its rest frame (the S' frame) then the angular distribution of photons in an annulus of solid angle between θ' and $\theta' + {\rm d}\theta'$ will be given by

$$P(\theta')\ {\rm d}\theta' = \frac{1}{2}\ \sin\,\theta'\ {\rm d}\theta'.$$

(Recall this from you old kinetic theory notes if you have forgotten it!). Use the relativistic abberation formulae to transform θ' to the observer's (S) frame θ and hence show that the distribution of photons in the S frame is given by

$$P(\theta)\ {\rm d}\theta = D^2\frac{1}{2}\ \sin\,\theta\ {\rm d}\theta,$$

where D is the relativistic Doppler factor, $[\gamma(1-\beta{\rm cos}\theta)]^{-1}$.

Problem 8: Calculation for beamed jets

Detailed study of the (single-sided) jet in the quasar 3C273 suggests that the jet is pointing towards us, at about 10° from the line-of-sight, has a bulk Lorentz factor $\gamma = 11$, and a synchrotron spectral index of 0.5. The deepest radio maps of 3C273 have a dynamic range of about 5000 and show no sign whatsoever of a counterjet. Is this surprising?

Problem 9: Inverse-Compton scattering (1999 finals)

Explain what is meant by *Thomson scattering* and *inverse Compton scattering*. Give **two** examples where inverse Compton scattering is important in astrophysics.

The power P in scattered radiation due to Thomson scattering is given by

$$P = c\sigma_{\mathrm{T}}U$$

where σ_{T} is the Thomson scattering cross-section and U is the energy density of incident radiation. Use this relation to derive an expression for the average power of radiation inverse Compton scattered by a population of relativistic electrons with Lorentz factors $\gamma \sim 1000$. Assume that the distribution of electrons is isotropic and that the radiation being scattered is at radio frequencies. Justify each step in your derivation.

Ultra-relativistic electrons of energy $10^{11}\,\mathrm{eV}$ are observed at the Earth. By considering the effect on the electrons of inverse Compton scattering of the microwave background radiation, calculate the maximum length of time for which the electrons could have had energies larger than the observed value. Given that the age of the Galaxy is $\sim 10^{10}$ years, what does this imply? Suggest some possible sources of such high-energy electrons.
[The energy density of the microwave background is $2.6 \times 10^{5}\,\mathrm{eV\,m^{-3}}$; $\sigma_{\mathrm{T}} = 6.6 \times 10^{-29}\,\mathrm{m^{2}}$.]