Stellar Atmospheres & Radiative Transfer

Patrick Roche

The Sun as a Typical Star

- Solar Interior cannot be probed directly (except neutrinos & helioseismology)
- Emerging radiation from solar atmosphere tells us
  - Total Flux (Luminosity)
  - Photospheric Temperature, Density
  - Surface Abundances
  - Dynamics
- Photosphere well defined layer
  - \( T \approx 5800 \text{K}, \rho \approx 10^{-4} \text{ kg m}^{-3} \)
  - \( \sim 300 \text{ km thick}, (0.0005 R_s) \)
  - Fraunhofer Spectrum
  - Limb Darkening, Granulation, Sunspots.

Stellar Spectra

- Stars classified via spectral lines
- HD Spectral Sequence OBAFGKM 40,000 -> 1500 K
- Sun is a G2V main sequence star (V= dwarf, high surface gravity)
- Fraunhofer absorption lines - element abundances seen against continuous (approx black body) spectrum
- Variability (sun spot cycle, rotation, pulsation, flares)

Stars

- Most stars appear as point sources to us
  - except some nearby supergiants
  - interferometric observations - asymmetries, limb darkening
  - doppler tomography eclipse/ rotation timing, occultations
- Measure integrated properties, averaged over stellar surface - simplified models
- Photosphere is thin compared to stellar diameter - approximate to plane parallel atmosphere, assume steady state
- Bolometric luminosity can be estimated, but need to know distance, extinction from \( T_{\text{eff}} \) can infer stellar diameter

Effective temperatures of stars

Neglecting interstellar absorption, the total energy arriving above the Earth’s atmosphere is its observed flux, \( f \), corrected for the distance to the star, i.e. \( L = 4\pi f d^2 \)

The same energy must be emitted by the star, i.e. \( L = 4\pi R^2 F \) where \( F \) is the surface flux, so \( F = f (d/R)^2 \). For the Sun, the angular radius is 960 arcsec and \( f = 3.17 \times 10^{10} \text{ erg/cm}^2/\text{s} \) so radiant flux at surface is \( F = 6.32 \times 10^{10} \text{ erg/cm}^2/\text{s} \).

The Stefan-Boltzmann law, \( F = \sigma T^4 \), or alternatively \( L/(4\pi R^2) = \sigma T^4 \), defines the ‘effective temperature’ of a star, i.e. the temperature which a black body would need to radiate the same amount of energy as the star, which is 5777K for the Sun.

IR Flux method for estimating stellar diameters: gives angular diameters, which can be converted to physical diameters if the distance to the star can be established.
Radiation Intensity, Energy and Flux

- Particular astronomical definitions
- Specific Intensity \( I_\nu \) (Wm\(^{-2}\) Hz\(^{-1}\) sr\(^{-1}\))
  or \( I_\lambda \) (Wm\(^{-2}\) \(\mu\)m\(^{-1}\) sr\(^{-1}\))

Brightness of an extended radiating surface

Note that it is independent of distance but also that it can only be measured directly from a resolved surface

The amount of energy per unit bandwidth at position \( r \) passing through unit area normal to the beam into solid angle \( d\Omega \) is

Energy flow is related to \( I \) by

\[
dtd\Omega = u_\nu \cos \theta dud\Omega dt
\]

where \( d\text{Area} \) is the projected area, and no azimuthal \( \phi \) dependence

Mean Intensity

The mean intensity \( I_\nu \) is the directional average (over 4\pi steradians) of the specific intensity.

\[
J_\nu = \frac{1}{4\pi} \int I_\nu d\Omega = \frac{1}{2} \int_0^{\pi} I_\nu \sin \theta d\theta
\]

For isotropic radiation, e.g. at the centre of a star, \( J_\nu = I_\nu \)

Energy density \( U_\nu = (4\pi/c) J_\nu \)

= radiation energy/ volume

where volume = c dt dA cos\( \theta \)

Flux

The flux, \( F_\nu \) (in Wm\(^{-2}\) Hz\(^{-1}\)) is the projection of the mean intensity in the radial direction, integrated over all solid angles. It is the net rate of energy flow per unit area. At a stellar surface, the flow is essentially all radially outwards

\[
F_\nu = \int \frac{dE_\nu}{dA d\nu dt} = \int I_\nu \cos \theta d\Omega
\]

Flux falls with the square of the distance and is the quantity measured for almost all stars

A simple example

Let us consider a point on the boundary of a radiating sphere. From above,

\[
F_\nu = \int_0^{\pi} \frac{d\phi}{dE_\nu} \int I_\nu \sin \theta \cos \theta \theta d\theta d\phi
\]

If no flux enters the surface and if there is no azimuthal dependence for \( F_\nu \)

\[
F_\nu = 2\pi \int _0^{\pi} I_\nu \sin \theta \cos \theta d\theta d\phi
\]

If \( I_\nu \) is independent of direction over one hemisphere then \( F_\nu = \pi I_\nu \) and \( J_\nu = \frac{1}{2} I_\nu \).

Moments of the Radiation Field

In a 1-dimensional geometry, the moments are:

- 0-th moment: \( J_\nu = \frac{1}{2} \int I_\nu \mu d\mu \)
  = Mean Intensity
- 1st moment: \( H_\nu = \frac{1}{2} \int I_\nu \mu d\mu \)
  = Eddington Flux
- 2nd moment: \( K_\nu = \frac{1}{2} \int I_\nu \mu^2 d\mu \)
  = K-Integral
- n-th moment: \( \mu^n \int I_\nu \mu^n d\mu \)

where \( \mu = \cos \theta \) and \( H_\nu \) is known as the Eddington Flux

Radiation Pressure

- A photon has momentum \( h/\nu \) where \( c \) is the velocity of light in the direction of propagation. Net transfer of momentum from photons impacting on an area \( dA \) produces a Pressure \( P \)
- Momentum transfer = \( d\mathcal{E}/c = I_\nu \cos \theta d\Omega dA dt/c \)
- Net transfer of normal component of momentum across \( dA \) by radiation of bandwidth \( d\nu \) is

\[
\frac{dAdt}{c} \int I_\nu \cos \theta d\Omega d\nu
\]

integrated over the sphere

giving pressure

\[
\nu = \frac{2\pi}{c} \int I_\nu \cos \theta d\theta = \frac{4\pi}{c} K_\nu = \frac{1}{3} U_\nu
\]
Measurement of Stellar Flux

Annulus on stellar surface has an area $2\pi r \, dr = 2\pi R^2 \sin \theta \cos \theta \, d\theta$ normal to line of sight and subtends solid angle $d\Omega = 2\pi (R/D)^2 \sin \theta \cos \theta \, d\theta$ with distance $D$.

Flux measured by observer is $F_\nu = (R/D)^2 F_\nu(0)$.

Emission and Absorption

So far have considered free radiation transport, but in real stars, photons are absorbed and scattered (absorbed energy may be thermalised before being re-emitted).

Consider radiation of Specific Intensity $I_\nu$ and solid angle $d\Omega$ normally incident on a slab of stellar atmosphere with cross section $dA$, thickness $ds$ and density $\rho$.

As it propagates through, the beam loses energy through absorption

$$dE_\nu = \kappa_\nu I_\nu \rho \, ds \, dA \, d\Omega \, d\nu$$

where $\kappa_\nu$ is the extinction coefficient per unit mass, or opacity, and consists of scattering and absorption terms $\kappa_\nu = \sigma + \alpha$.

Radiative Transfer

Energy emitted in the direction of propagation

$$dE_\nu = J_\nu I_\nu \rho \, ds \, dA \, d\Omega \, d\nu$$

where $J_\nu$ is the emission coefficient per unit mass, containing contributions from scattering and thermal emission.

The ratio $J_\nu / \kappa_\nu$ is known as the Source Function denoted by $S_\nu$.

In general $\kappa_\nu$ will vary as a function of $x$, and the integral of the extinction coefficient w.r.t. distance is the Optical Depth, $\tau$.

$$\tau = \int k(x) \, dx$$

and Specific Intensity falls off with $\tau$ as:

$$I = I_0 e^{-\tau}$$

and optical depth $\tau = \ln (I/I_0)$ (note: $\tau$ is measured inwards)

The difference between energy emitted and absorbed in the element is related to the change in Specific Intensity of the beam:

$$dI_\nu / dA \, d\Omega \, d\nu = (J_\nu \rho \, ds - \kappa_\nu I_\nu \rho \, ds) \, dA \, d\Omega \, d\nu$$

Optical Depth

And with the path through the slab $ds = dz / \cos \theta = dz / \mu$

$$\kappa_\nu \rho \, d\nu$$

In general $\kappa_\nu$ will vary as a function of $x$, and the integral of the extinction coefficient w.r.t. distance is the Optical Depth, $\tau$.

$$\tau = \int k(x) \, dx$$

and Specific Intensity falls off with $\tau$ as:

$$I = I_0 e^{-\tau}$$

and optical depth $\tau = \ln (I/I_0)$ (note: $\tau$ is measured inwards)

Purely emitting medium: $\mu dI_\nu / dz = \rho J_\nu$

and no emission : $\mu dI_\nu / dz = -k_\nu \rho I_\nu$

Standard form of the Radiative Transfer Equation:

$$\mu dI_\nu / d\tau_\nu = I_\nu - S_\nu$$

Solution of the Radiative Transfer Equation

Second Order Differential Equation solved using an integrating factor $e^{\int \tau / \mu} \mu$

$$\frac{d}{d\tau} (I e^{-\tau/\mu}) = -S e^{-\tau/\mu}$$

Inserting boundary conditions:

$1(\mu) = I_0$ for $0 < \mu < 1$

$1(\mu) = 0$ for $\mu > 0$

Radiative transfer solution

The first term on the RHS represents the radiation transmitted by the medium, after passing through an optical depth $\tau$, the second term gives the contribution from the radiation emitted along the path.
Radiative Transfer
Consider radiation emerging normally ($\mu = 1$)

$$ I = I_0 e^{-\tau} + S(1 - e^{-\tau}) $$

Optically Thick case: $\tau >> 1$  $I = S$, see into the source to 1 optical depth
Optically Thin case: $\tau << 1$  $I = S\tau$, see into the whole volume

Note that $S$ is a function of the radiation field, and hence on the transfer equation - self-consistent solution may require iteration

Radiative Equilibrium
Energy generated in the core must emerge from the stellar surface - no sinks inside the star.

Energy Flux is Rigorously Conserved
i.e. $F_\nu = \int_0^\infty F_\nu(\sigma) \, d\nu$

Energy transport can occur via radiation and convection (conduction is usually unimportant)
If Radiation dominates will approach Radiative Equilibrium:
Energy removed from the beam via the opacity is replaced by emission.

$$ \int k_\nu J_\nu \, d\Omega \, d\nu = \int J_\nu \, d\Omega \, d\nu $$

or

$$ 4\pi \int k_\nu J_\nu \, d\nu = 4\pi \int k_\nu S_\nu \, d\nu $$

at each point in the atmosphere OR

Local Thermodynamic Equilibrium
A reasonable approximation when the conditions are close to the thermodynamic equilibrium values at the local Temperature and Pressure:
i.e. particle velocity distribution, emissivity, opacity and level populations of the gas approximate a thermal population, though the photon spectrum will probably not mimic a Planck distribution.
Under conditions of high density and/or high optical depth, it is possible to approach LTE (e.g. in stellar interiors or dense gas) - a reasonable approximation at the base of the photosphere where the continuum emission and weak lines arise.

Grey Atmosphere
Use frequency averaged opacity and LTE (may be OK for continuum but obviously does not work well for lines, physically frequency-independent opacity applies to electron scattering)

Recall that the radiation pressure is related to the 2nd moment of the Intensity

$$ \frac{2\pi}{c} I_\nu \int \cos^2 \theta d\theta = \frac{4\pi}{c} K_\nu $$

$$ K_\nu(\tau_\nu) = \frac{1}{4\pi} I_\nu \int \cos^2 \theta d\theta $$

Flux is conserved

$$ F_\nu = \int_0^\infty F_\nu \, d\nu $$

where

$$ F_\nu(0) = \int_0^\infty F_\nu \cos^2 \theta d\theta $$

and integrating over solid angle and frequency

and taking $\bar{k}$ as the frequency averaged opacity

$$ \frac{dK_\nu}{d\tau_\nu} = \frac{F(0)}{4\pi} \bar{k} $$

Differentiating then gives

$$ \frac{dK_\nu}{d\tau_\nu} = \frac{F(0)}{4\pi} \bar{k} = J_\nu - S_\nu = 0 $$

The Eddington Approximation
Integrating gives

$$ \frac{1}{4\pi} \frac{d}{d\tau_\nu} \left( \tau_\nu K_\nu \right) = \left( \frac{d}{d\tau_\nu} \right) $$

where $I = J$ and

$$ d\sigma_\nu / d\Omega (\nu) = J_\nu(\nu, \theta) \cos \theta d\theta d\nu = J_\nu(\nu, \theta) \cos \theta d\Omega $$

which is known as the Eddington approximation.

Applying this gives

$$ \frac{d}{d\tau_\nu} \left( \tau_\nu K_\nu \right) = \frac{d}{d\tau_\nu} (F(0)) $$

and

$$ \frac{d}{d\tau_\nu} \left( \tau_\nu K_\nu \right) = \frac{d}{d\tau_\nu} (F(0)) $$
The Eddington Approximation

Now in a grey atmosphere, the source function $S(\tau) = J(\tau) = \frac{2}{\pi} \tau + \text{constant}$

Consideration of the boundary conditions:

at the surface, there is no flux inward, i.e. $I_\nu(0, \theta) = 0$ for $-1 < \mu < 0$ and $I_\nu(\theta)$ is the intensity in the outward direction $P_1$ for $0 < \mu < 1$

$I_\nu(0, \theta) = \frac{1}{2 \pi} F(\theta)$

and the constant of integration is $S(0) = \frac{1}{2 \pi} F(0)$

and in LTE, the source function $S(\tau) = B(\tau) = \frac{\sigma T^4}{\pi}$

and $F(0) = \sigma T^4$

Note that when $\tau = 2/3$, $T = T_{\text{eff}}$; so that the effective depth at which the continuum is emitted is $\tau = 2/3$

Limb Darkening

As the line of sight moves from the centre to the edge of the stellar disk, it passes through an increasing path length of atmosphere.

Degree of limb darkening depends on the optical depth, as $\tau$ increases, it approximates an opaque surface, with a hard edge, so see into very similar physical depths at centre and edge.

Opacity

Contributions to opacity:

– Hydrogen is the dominant species
– Bound-free absorption:
  • Balmer edge (ionization limit from $n=2$) occurs at 365 nm so cannot contribute to visible opacity
  • Paschen edge (ionization limit from $n=3$) occurs at 820 nm so will contribute opacity in the visible, but with $T < 6000 K$, the population in the $n=3$ level is very small (Saha equation)

Free-free absorption will also give a continuum opacity, but at a low level.

Hydrogen can form a negative ion with a proton + 2 electrons. The dissociation energy of $\text{H}^-$ is 0.75 eV (1.65 $\mu m$) and so it can provide continuum opacity in the visible and near-infrared.

$\text{H}^-$ will dominate in cool stars, but with increasing photospheric temperature, higher $n$ levels in H will be populated and atomic H bound-free dominates in A-type stars.

Bound-bound transitions may be important. H has few transitions, but some metals (e.g., Fe) have many.

In cool stars, molecules dominate the spectrum TiO, VO, CO, C$_2$
LTE Level Populations

Ionization equilibrium described by the Saha equation
Atomic/ionic levels populated thermally (Boltzmann distribution)
Ion and e⁻ velocity distributions are Maxwellian
The source function is given by the Planck function.

Boltzmann Equation
\[ \frac{N_i}{N_j} = \frac{g_i e^{-\frac{\chi_i}{kT}}}{g_j U_j(T)} \]

Where \( N_i \) is the population of level i with statistical weight \( g_i \) and excitation energy \( \chi_i \) of ionization stage j; \( N_j \) is the total population and \( U_j(T) \) is the partition function.

In the solar photosphere, the fraction of H excited to n=2 is \( \sim 10^{-8} \).

Line Spectra

Narrow Spectral lines arise from bound-bound transitions and can be in either emission or absorption (or both).
They provide additional opacity at specific frequencies.
Strong lines have high opacities and are formed higher in the atmosphere where T is usually lower, so appear in absorption.

Calculate the emergent profile across the line from the line absorption coefficient \( l_0 = N_i \alpha \), where \( \alpha \) is the atomic absorption coefficient.

So now have \( k_\nu = l_0 + k_c \cdot k \) as \( k \) is constant over line width.

The Line Spectrum

A spectral line is described by its profile. The line depression, \( D_\nu \), compares the intensity in the line with the nearby continuum. The equivalent width is the integral of \( D_\nu \) and is the same width as a rectangular piece of spectrum that blocks the emergent intensity:

\[ W = \int D_\nu \, d\nu \]

Line Broadening Mechanisms

Lines are not sharp due to some or all of the following broadening mechanisms:
Natural or radiation damping: lifetimes of excited states
Collisional: collisions or perturbations by other particles
Rotational: stellar rotation
Turbulent: mass motions in atmosphere
Zeeman splitting: magnetic effects

Broadening

- Natural broadening
  - lifetime in level \( u : \Delta = 1/A_u \) (Einstein A coefficient)
  - Energy Spread (Heisenberg) \( \Delta E = h/(2\pi\Delta) \)
  - Natural broadening : Lorentz Profile
- Collisional Broadening - (dwarf stars)
  - Excitation, de-excitation through collisions
  - Collision lifetime of level \( u = t_c \)
  - Energy spread \( t_c \Delta E = h \)
- Doppler
  - \( \Delta \nu/\nu = v/c \) component along line of sight (+ rotation)
    (classical)

Curve of Growth

- As optical depth increases, EW initially increases linearly, but
- Line core saturates before wings, and EW grows more slowly

\( t = 1/16, 1/8, 1/4 \ldots 2,4,8 \)
Detailed shape depends on line profile Gaussian, Lorentzian etc
Can calibrate optically thick lines to extract atomic column

Voigt profile
Voigt profiles (Doppler cores, pressure Lorentz wings) of the K line of Ca II. The shallowest line is produced by $N = 3 \times 10^{11}$ ions/cm$^2$, and the ions are ten times more abundant for each successive profile.
Note that line cores are not completely black - still see emitting surface at $\tau \sim 1$

Curve of Growth
- The Curve of growth describes how $W_\lambda$ depends on the number of absorbing atoms or ions –
- For weak, optically thin lines, as the abundance doubles, the line equivalent width also doubles in strength: $W_\lambda \sim N_s$ – this is the LINEAR part of the curve of growth
- As the abundance continues to increase, the Doppler core of the line becomes optically thick and saturates. The wings of the line, which are still optically thin, deepen, which occurs with little change in the line equivalent width and so produces a PLATEAU in the curve of growth, $W_\lambda \sim (\ln N_s)^{1/2}$.
- Ultimately, the pressure broadening contributes to the wings, strengthening the equivalent width, $W_\lambda \sim (N_s \gamma f)^{1/2}$ . This is the DAMPING or SQUARE ROOT part of the curve of growth.

Analysis
- Using the curve of growth and a measured equivalent width we can derive the number of absorbing atom,
- The Boltzmann and Saha equations convert this value into the total number of atoms of that element in the photosphere.
- Locate several lines on a curve of growth to reduce errors.
- Beware of complications - e.g. emission cores in absorption lines