Accretion in Astrophysics: Theory and Applications

Solutions to Problem Set I

(Ph. Podsiadlowski, SS10)

1 Luminosity of a Shakura-Sunyaev (SS) Disk

In lecture we derived the following expression for the effective temperature, $T_{\rm eff}$ as a function of radial distance from the central compact star:

$$T_{\rm eff} = \left[\frac{3GM\dot{M}}{8\pi\sigma r^3}\right]^{1/4} \left(1 - \sqrt{r_0/r}\right)^{1/4}$$

where σ is the Stefan-Boltzmann constant.

a.) The total power radiated by the disk (including both sides) is given by:

$$L = 2 \times \int_{r_0}^{\infty} \sigma T_{\rm eff}^4 \; 2\pi r \; dr = \int_{r_0}^{\infty} \frac{3GM\dot{M}}{2r^3} \left(1 - \sqrt{r_0/r}\right) r \; dr$$

This is easily integrated analytically to yield:

$$L = \frac{1}{2}\frac{GM\dot{M}}{r_0}$$

b.) Define the power radiated in an SS disk for all radii greater than r to be $L(> r)$.

Solution: The indefinite integral for $L(> r)$ is:

$$L(> r) = \frac{3GM\dot{M}}{2}\left[\frac{1}{r} - \frac{2\,r_0^{1/2}}{3\,r^{3/2}}\right]$$

The ratio of this quantity to the gravitational energy release is

$$\frac{L(> r)}{\frac{1}{2}\frac{GM\dot{M}}{r}} = \left[3 - 2\sqrt{\frac{r_0}{r}}\right]$$

Sketch the ratio as a function of r. This result demonstrates that the gravitational potential energy, released as the matter migrates inward, does not emerge from the disk locally, but rather is redistributed by the viscous stresses.

2 Temperature of an SS-Accretion Disk

a.) *Use the above expression for $T_{\rm eff}$ of an SS-disk to find the location (i.e., the radial distance from the central star) where the temperature is a maximum. Express your answer in terms of r_0, the radius of the inner edge of the disk. If the central star is a non-rotating black hole, then $r_0 = 6R_g$. In this case, express your answer for the location of the maximum temperature in terms of R_g.*

Solution: The disk temperature can be written as:

$$T_{\rm eff} = T_0 \, \left(\frac{r_0}{r}\right)^{3/4} \left(1 - \sqrt{r_0/r}\right)^{1/4}$$

where $T_0 \equiv (3GM\dot{M}/8\pi\sigma r_0^3)^{1/4}$. Differentiating the function of r in the above expression, and equating the result to 0, yields the radius where the maximum in the disk temperature occurs:

$$r(T_{\rm max}) = \frac{49}{36}\, r_0$$

The maximum value of the disk temperature is found by evaluating T at $49\, r_0/36$ We find

$$T_{\rm max} = T_0 \, \frac{6\sqrt{6}}{7^{7/4}} \simeq 0.488 \; T_0$$

b.) *Compute $T_{\rm max}$ for the following types of accreting sources:*

accretor	mass	$\dot{M}$ (gm/sec)	r_0 (cm)	source type	T_0 (K)	$T_{\rm max}$ (K)
white dwarf	1 $M_\odot$	10^{17}	9×10^8	"CV"	7.9×10^4	3.9×10^4
neutron star	1.4 $M_\odot$	10^{18}	1.2×10^6	"LMXB"	2.1×10^7	1.0×10^7
black hole	10^6 $M_\odot$	10^{24}	9×10^{11}	"AGN"	7.9×10^5	3.9×10^5
black hole	10^9 $M_\odot$	10^{27}	9×10^{14}	"AGN"	1.4×10^5	6.8×10^4

3 Mass Stored in an Accretion Disk

In lecture we derived expressions for the midplane pressure, temperature, and density of an SS-disk, as well as for the thickness, H, all as functions of the radial distance r. In the handout, the dependence of these quantities on α and $\dot{M}$ were specified, but the leading dimensioned quantities were not given. These are provided below for the case of an accreting central neutron star with a mass of 1.4 $M_\odot$.

Use these results to compute the amount of mass stored in the accretion disk at a particular instant in time. Formally, you will find that this mass is infinite; however, if you restrict yourself to plausible integration limits for r, e.g., $r_0 < r < 10^4\ r_0$, you will find a sensible answer.

$$\begin{aligned}
P &\simeq 2\times 10^5 \alpha^{-9/10} \dot{M}_{16}^{17/20} r_{10}^{-21/8} f^{17/20} \qquad \text{dynes cm}^{-2} \\
H &\simeq 1\times 10^8 \alpha^{-1/10} \dot{M}_{16}^{3/20} r_{10}^{9/8} f^{3/20} \qquad \text{cm} \\
T &\simeq 2\times 10^4 \alpha^{-1/5} \dot{M}_{16}^{3/10} r_{10}^{-3/4} f^{3/10} \qquad \text{K} \\
\rho &\simeq 7\times 10^{-8} \alpha^{-7/10} \dot{M}_{16}^{11/20} r_{10}^{-15/8} f^{11/20} \qquad \text{g cm}^{-3}
\end{aligned}$$

where $\dot{M}_{16}$ *is the mass accretion rate in units of* 10^{16} *gm sec*$^{-1}$*, and* r_{10} *is the radial distance in units of* 10^{10} *cm. The function* f *is defined to be* $f = (1 - \sqrt{r_c/r})^{1/4}$*.*

To make the integration easier, but with no significant loss of accuracy, you can safely set $f = 1$ *in the above expressions. Take the inner edge of the accretion disk to be located at* $r_0 = 10^7$ *cm, and the accretion rate to be* $\dot{M} = 10^{18}$ *grams sec*$^{-1}$*. A plausible value to use for the* α *parameter is 0.1.*

Solution: The mass in the disk, at any given time, is just the integral of density over the volume.

$$M_{\rm disk} \simeq \int_{r_0}^{r_{\rm max}} \rho(r)\; 2\pi\, r\, H(r)\, dr$$

The volume element of an annulus is $dV = 2\pi r H dr$. Next, we plug in the given expressions for ρ and H:

$$M_{\rm disk} \simeq \int_{r_0}^{r_{\rm max}} 7\alpha^{-4/5} \dot{M}_{16}^{7/10} r_{10}^{-3/4} f^{7/10}\, 2\pi\, r\, dr = 14\pi\alpha^{-4/5} \dot{M}_{16}^{7/10} \int_{r_0}^{r_{\rm max}} r_{10}^{-3/4} f^{7/10}\, r\, dr$$

This integral, if carried out with the factor $f^{7/10}$ included, yields a hypergeometric series, which is not particularly insightful. To obtain much better than an order-of-magnitude estimate, we simply set $f = 1$, and complete the integral to find:

$$M_{\rm disk} \simeq 14\pi\alpha^{-4/5} \dot{M}_{16}^{7/10} \int_{r_0}^{r_{\rm max}} r_{10}^{-3/4}\, r\, dr \;\;\simeq\;\; 14\frac{4}{5}\pi 10^{20} \alpha^{-4/5} \dot{M}_{16}^{7/10} (r_{\rm max,10}^{5/4} - r_{0,10}^{5/4})$$

where the factor of 10^{20} comes from converting r to r_{10}. If $r_{\rm max} \gg r_0$, then our expression for the mass in the disk is:

$$M_{\rm disk} \simeq 14\frac{4}{5}\pi 10^{20} \alpha^{-4/5} \dot{M}_{16}^{7/10} r_{\rm max,10}^{5/4}$$

For the parameters specified in the problem: $\alpha = 0.1$, $\dot{M}_{16} = 100$, and $r_{\rm max} = 10^4 r_0 = 10^{11}$ cm, the mass in the disk is:

$$M_{\rm disk} \simeq 9.9\times 10^{24} \;\;\text{grams}$$

Note that this is very much less than the mass of the central neutron star.

Given the amount of mass stored in such a disk and the accretion rate, estimate a timescale for "filling" the disk if it were initially empty.

Solution: The disk filling timescale is

$$\tau_{\rm fill} \simeq \frac{M_{\rm disk}}{\dot{M}} \simeq \frac{9.9\times 10^{24}\ \text{grams}}{10^{18}\ \text{grams s}^{-1}} \simeq 10^7\ \text{sec} \simeq 4\ \text{months}$$

4 Radial Velocity in an SS Accretion Disk

Solution: Use the expressions for $\rho(r)$ and $H(r)$ given in the previous problem to compute an expression for v_r, the radial in-spiral speed of the disk material. Show that for all choices of parameters α and $\dot{M}$, the radial speed $v_r \ll v_{\rm kepler}$, as long as one considers radial distances significantly greater than r_0.

From our conservation of mass expression, we found:

$$\dot{M} = \rho H 2\pi r v_r$$

where v_r is the radial flow velocity of the material in the accretion disk. Solving for v_r we have

$$v_r = \frac{\dot{M}}{\rho H 2\pi r}$$

We now use the results from Problem #3 for the quantity $\rho H 2\pi r$ to show that

$$v_r = \frac{\dot{M}}{7\alpha^{-4/5}\dot{M}_{16}^{7/10} r_{10}^{-3/4} f^{7/10}\, 2\pi\, r}$$

For radial distances $\gg r_0$ we can safely set $f = 1$. Let us work with a dimensionless ratio, v_r/v_K:

$$\frac{v_r}{v_K} \simeq \frac{\dot{M}\sqrt{r}}{7\alpha^{-4/5}\dot{M}_{16}^{7/10} r_{10}^{-3/4}\, 2\pi\, r\sqrt{GM}}$$

where the square-root terms represent $1/v_K$. Since the units of $\dot{M}$ and r are mixed, this needs to be tidied up:

$$\frac{v_r}{v_K} \simeq \frac{10^{16}10^{-5}\alpha^{4/5} r_{10}^{1/4}\dot{M}_{16}^{3/10}}{7 \cdot 2\pi\sqrt{GM}}$$

Finally, we express the mass of the neutron star in units of 1.4 $M_\odot$, and evaluate all the constants:

$$\frac{v_r}{v_K} \simeq 0.00017\, \frac{\alpha^{4/5} r_{10}^{1/4}\dot{M}_{16}^{3/10}}{\sqrt{M/1.4\, M_\odot}}$$

The α parameter is likely to be considerably less than unity, but we can obtain an upper limit to v_r/v_K by setting $\alpha = 1$. In this case, even if $\dot{M}_{16}$ is near the Eddington limit of $\sim$100, and r_{10} is as large as 100, we see that

$$\frac{v_r}{v_K} < 0.002$$

5 Spectrum of an SS Accretion Disk

Write out an integral expression for L_ν of an SS accretion disk, where L_ν is the spectral luminosity (units of power per unit frequency interval). Treat each annulus in the disk as a black body of temperature $T_{\rm eff}(r)$ as defined in problem 1 above. Do not try to integrate the expression since it can't be done analytically.

For reference, the Planck function is:

$$P(\nu) = \frac{2\pi h\nu^3 c^{-2}}{\left[e^{(h\nu/kT)} - 1\right]}$$

Solution: The disk spectrum can be obtained by integrating the Planck function over the surface area of the disk, taking into account the fact that T varies with radial distance:

$$L_\nu = \int_{r_0}^{\infty} \frac{2\pi h\nu^3 c^{-2}}{\left[e^{[h\nu/kT(r)]} - 1\right]}\, 2\pi r\, dr = 4\pi^2 h\nu^3 c^{-2} \int_{r_0}^{\infty} \frac{r}{\left[e^{[h\nu/kT(r)]} - 1\right]}\, dr$$

where this is the spectrum emanating from one side of the disk (the units are energy time^{-1} frequency^{-1}).

<u>*Optional*</u>

If you make the following approximations, the spectrum (i.e., L_ν) can be obtained analytically:
- *Approximate the Planck function by*

$$P(\nu) = 2\pi h\nu^3 c^{-2} e^{-h\nu/kT}$$

- *Take the factor $\left(1 - \sqrt{r_0/r}\right)^{1/4}$ in the expression for $T(r)$ to be approximately unity.*
- *Carry out the integration from $r = 0$ to $r = \infty$, even though a real disk obviously has limits at both ends.*

Solution: If we utilize the proposed approximation to the Planck function in the above integral, we have:

$$L_\nu = \int_{r_0}^{\infty} 2\pi h\nu^3 c^{-2} e^{-[h\nu/kT(r)]}\, 2\pi r\, dr$$

where

$$T_{\rm eff} = \left[\frac{3GM\dot{M}}{8\pi\sigma r^3}\right]^{1/4} \left(1 - \sqrt{r_0/r}\right)^{1/4}$$

If we now set the factor $\left(1 - \sqrt{r_0/r}\right)^{1/4}$ in the expression for $T(r)$ to be approximately unity, as suggested, we can write:

$$T_{\rm eff} \simeq \left[\frac{3GM\dot{M}}{8\pi\sigma r_0^3}\right]^{1/4} \left(\frac{r_0}{r}\right)^{3/4} \equiv T_0 \left(\frac{r_0}{r}\right)^{3/4}$$

The integral for the spectral luminosity now becomes:

$$L_\nu = 4\pi^2 h\nu^3 c^{-2} \int_{r_0}^{\infty} e^{-[(h\nu/kT_0)(r/r_0)^{3/4}]}\, r\, dr$$

With an appropriate change of variable, this integral is analytic, and we find:

$$L_\nu \simeq 4\pi^2 h\nu^3 c^{-2}\, \frac{4}{3}\, \Gamma\left(\frac{8}{3}\right) r_0^2 \left(\frac{kT_0}{h\nu}\right)^{8/3}$$

Collecting the terms in frequency, ν, we can get an idea of the spectral shape of the disk:

$$L_\nu \simeq 4\pi^2 h c^{-2}\, \frac{4}{3}\, \Gamma\left(\frac{8}{3}\right) r_0^2 \left(\frac{kT_0}{h}\right)^{8/3} \nu^{1/3} \;\propto\; \nu^{1/3}$$

6 The Last Stable Circular Orbit

The equation for the radial coordinate r of a test particle of non-zero rest mass orbiting a non-rotating black hole of mass M (the radial geodesics equation) is

$$\frac{1}{2}\dot{r}^2 + \frac{1}{2}\left(1 - \frac{2GM}{c^2 r}\right)\left(\frac{L^2}{r^2} + c^2\right) = \frac{1}{2}\frac{E^2}{c^2}. \tag{1}$$

To illustrate its similarity to the Newtonian energy conservation equation, one can also write this as

$$\underbrace{\frac{1}{2}\dot{r}^2 - \frac{GM}{r} + \frac{L^2}{2r^2}}_{\text{Newtonian}} - \frac{GML^2}{c^2 r^3} = \frac{1}{2}\frac{E^2 - c^4}{c^2} = E'.$$

For the purposes of any calculation, it is useful to introduce non-dimensional variables, in particular

$$x \equiv \frac{rc^2}{GM}, \quad l \equiv \frac{Lc}{GM}, \quad e \equiv \frac{E}{c^2}, \quad \tau \equiv \frac{tc^3}{GM}. \tag{2}$$

Then, equation (1) can be written as

$$\frac{1}{2}x'^2 + \frac{1}{2}\left(1 - \frac{2}{x}\right)\left(\frac{l^2}{x^2} + 1\right) = \frac{1}{2}e^2 = e', \tag{3}$$

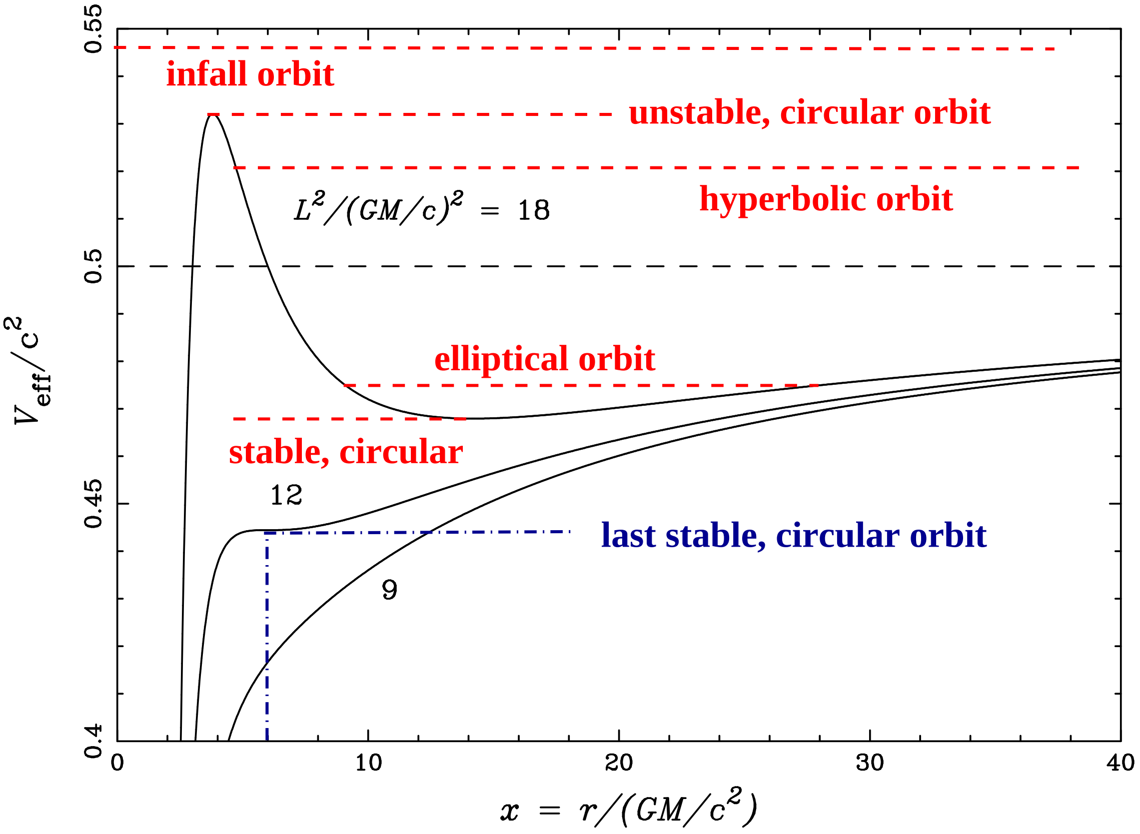
where now $x' = dx/d\tau$. Note that the energy at infinity for a non-moving particle is $e = 1$ (or $e' = 1/2$), due to its rest mass energy.

Part a)

By defining an effective potential $V_{\text{eff}} = 1/2\,(1 - 2GM/c^2 r)\,(L^2/r^2 + c^2)$, one can classify the possible trajectories/orbits just as in first-year classical mechanics (keeping in mind that the 'integration constant' $(1/2\,E^2/c^2)$ at infinity is $1/2\,c^2$ for a non-moving particle at infinity [due to the rest mass energy]).

The figure shows the non-dimensional effective potential for various values of L^2, as indicated. If $L^2 < 12G^2M^2/c^2$ (see next part), the effective potential has no maxima and minima, and hence particles of any energy can fall into the black hole.

If $L^2 > 12G^2M^2/c^2$, there is one maximum and one minimum in the effective potential, and the particle trajectories depend on the energies. If e' is larger than the value at the maximum potential, all particles with an inward radial velocity will fall into the black hole. If the energy is equal to this maximum, the corresponding orbit is an *unstable circular orbit*, if e' is less than the maximum, but larger than $1/2$, the value of e' for a stationary particle at infinity, the particle is on a *hyperbolic trajectory* (approaching the black hole to a minimum separation where e' equals the effective potential, the turning point). If e' is less than $1/2$, but larger than the potential at the minimum, the particle is trapped on an *elliptical orbit* between the two turning points of the potential with $e' = V_{\text{eff}}/c^2$. A particle with e' equal to the minimum of V_{eff}/c^2 corresponds to a stable, circular orbit.



Part b)

In non-dimensional units (to get simpler expressions) the effective potential can be written as

$$\hat{V}_{\text{eff}}(x,l) = \frac{1}{2}\left(1-\frac{2}{x}\right)\left(\frac{l^2}{x^2}+1\right). \tag{4}$$

Circular orbits require $\mathrm{d}\hat{V}_{\text{eff}}/\mathrm{d}x = 0$:

$$\frac{\mathrm{d}\hat{V}_{\text{eff}}}{\mathrm{d}x} = \frac{1}{x^2} - \frac{l^2}{x^3} + \frac{3l^2}{x^4} = 0,$$

which leads to a quadratic equation with the solution

$$x_{\pm} = \frac{l^2 \pm \sqrt{l^4 - 12l^2}}{2}$$

(as given on the sheet). Real solutions only exist when the $\sqrt{}$ is real, i.e. $l^2 > 12$.

Part c)

The x_+ (or r_+) solution as a function of l^2 has an obvious minimum when the $\sqrt{}$ expression is zero, i.e. when $l^2 = 12$, where the solution is $x_+^{\rm min} = 6$ (or $r_+^{\rm min} = 6GM/c^2$). To show that the x_+ solution corresponds to a minimum, one could calculate the second derivative of $\hat{V}_{\rm eff}$. But this is not really necessary from the analysis of part a). Noting that for $l^2 > 12$ there will be *exactly* two extrema for $V_{\rm eff}$ (part b), that $r_+ \geq r_-$, and considering the asymptotic form of $V_{\rm eff}$ (part a), it follows that the r_+ solution must correspond to a minimum and the r_- solution to a maximum of the effective potential (i.e. correspond to a stable and unstable circular orbit, respectively). For the minimum r_+, obviously $r_+ = r_-$, and the effective potential has a saddle point.

Part d)

At the innermost stable orbit, $l^2 = 12$ and $x_+^{\rm min} = 6$ and hence (using equations 3 and 4)

$$\hat{V}_{\rm eff}^{\rm min} = \frac{4}{9} = \frac{1}{2}e^2,$$

which implies that the (non-dimensional) energy at this orbit is $e = \sqrt{8/9}$. Since the energy of a stationary particle at infinity is $e_\infty = 1$ (corresponding to its rest mass energy), the binding energy at that orbit is $e_{\rm B} = 1 - \sqrt{8/9}$ or in dimensional units (equation 2)

$$E_{\rm B} = (1 - (8/9)^{1/2})\, c^2 \simeq 0.06\, c^2.$$

Part e)

As matter slowly drifts radially inwards in an accretion disc (determined by the disc's viscosity), roughly half of the gravitational energy of the matter is released and radiated away (in the Newtonian case anyway). But once it reaches the innermost stable orbit, matter will rapidly fall into the black hole (i.e. pass through the event horizon) taking with it its total mass-energy (including all of its kinetic energy), adding to the mass of the black hole without further radiating away much of its energy. Therefore the binding energy at the innermost stable orbit limits the maximum energy that can be radiated away by an accreting black hole. This is different for a neutron star, which has no event horizon. Therefore, even if the last stable orbit were larger than the neutron star (possible for some equations of state), most of the kinetic energy would have to be released when matter impacts with the hard surface of the neutron star, providing a method of distinguishing, at least in principle, between an object with a hard surface and an event horizon (i.e. black hole).

Note: The analysis was done for a non-rotating black hole. If the black hole is rotating rapidly in the same direction as accreting matter, the innermost stable orbit is much closer and the accretion efficiency may be much larger (up to $\sim 30\,\%$ in realistic models). The argument also ignores pressure forces in the disc that become important when the accretion approaches the Eddington rate.