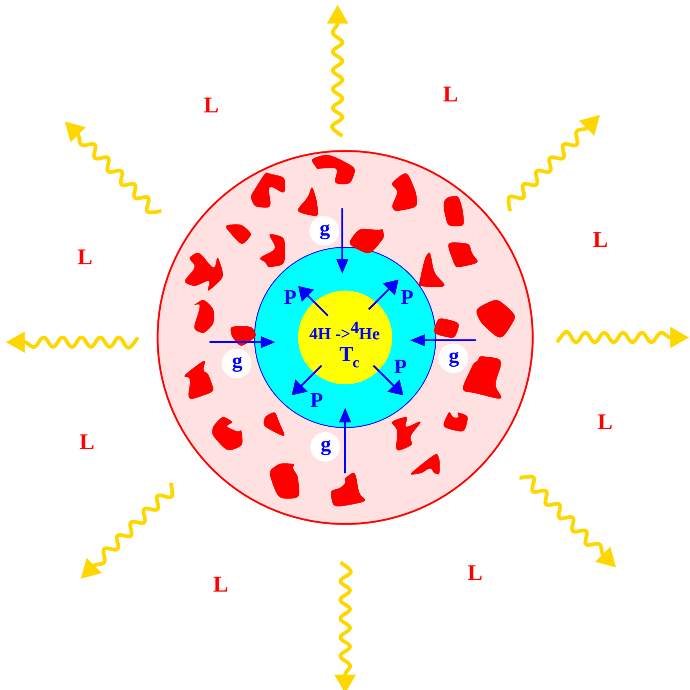


SUMMARY IV: FUNDAMENTAL PRINCIPLES

- Stars are *self-gravitating* bodies in *dynamical equilibrium* $\rightarrow$ *balance of gravity and internal pressure forces* (hydrostatic equilibrium);
- stars lose energy by *radiation from the surface* $\rightarrow$ stars supported by thermal pressure require an *energy source* to avoid collapse, e.g. *nuclear energy, gravitational energy* (energy equation);
- the *temperature structure* is largely determined by the mechanisms by which *energy* is *transported* from the core to the surface, *radiation, convection, conduction* (energy transport equation);
- the *central temperature* is determined by the *characteristic temperature* for the appropriate *nuclear fusion reactions* (e.g. H-burning: 10^7 K; He-burning: 10^8 K);
- normal stars have a *negative 'heat capacity'* (virial theorem): they heat up when their total energy decreases ($\rightarrow$ normal stars contract and heat up when there is no nuclear energy source);
- *nuclear burning is self-regulating* in non-degenerate cores (virial theorem): e.g. a sudden increase in nuclear burning causes expansion and cooling of the core: *negative feedback* $\rightarrow$ *stable nuclear burning;*
- the *global structure* of a star is determined by the *simultaneous satisfaction* of these principles $\rightarrow$ the *local properties* of a star are determined by the *global structure.*
  (Mathematically: it requires the simultaneous solution of a set of coupled, non-linear differential equations with mixed boundary conditions.)

4 THE EQUATIONS OF STELLAR STRUCTURE

In the absence of convection:

$$\frac{dP_r}{dr} = \frac{-GM_r\rho_r}{r^2} \quad (1)$$

$$\frac{dM_r}{dr} = 4\pi r^2\rho_r \quad (2)$$

$$\frac{dL_r}{dr} = 4\pi r^2\rho_r\left(\varepsilon_r - T\frac{dS}{dt}\right) \quad (3)$$

$$\frac{dT_r}{dr} = \frac{-3\kappa_r L_r\rho_r}{16\pi acr^2T_r^3} \quad (4a)$$

4.1 The Mathematical Problem (GZ: 16-2; CO: 10.5)

- $P_r, \kappa_r, \varepsilon_r$ are functions of ρ, T, chemical composition
- Basic physics can provide expressions for these.
- In total, there are *four, coupled, non-linear, partial differential equations (+ three constitutive relations) for seven unknowns:* $P, \rho, T, M, L, \kappa, \varepsilon$ as functions of r.
- These completely determine the structure of a star of given composition subject to boundary conditions.
- In general, only numerical solutions can be obtained (i.e. computer).
- *Four (mixed) boundary conditions* needed:
  - ▷ at centre: $M_r = 0$ and $L_r = 0$ at $r = 0$ (exact)
  - ▷ at surface: $L_s = 4\pi R_s^2\sigma T_{eff}^4$ (blackbody relation)
    (surface = photosphere, where $\tau \simeq 1$)
    $P = (2/3)\, g/\kappa$ (atmosphere model)

  (sometimes: $P(R_s) = 0$ [rough], but *not* $T(R_s) = 0$)

4.1.1 Uniqueness of solution: the Vogt Russell "Theorem" (CO: 10.5)

> *"For a given chemical composition, only a single equilibrium configuration exists for each mass; thus the internal structure of the star is fixed."*

- This "theorem" has not been proven and is not even rigorously true; there are known exceptions

4.1.2 The equilibrium solution and stellar evolution:

- If there is *no bulk motions* in the interior of a star (i.e. no convection), *changes of chemical composition* are *localised* in *regions of nuclear burning* The structure equations (1) to (4) can be supplemented by equations of the type:

$$\partial/\partial t\,(\text{composition})_M = f(\rho, T, \text{composition})$$

- Knowing the composition as a function of M at a time t_0 we can solve (1) to (4) for the structure at t_0. Then

$$(\text{composition})_{M,t_0+\delta t} = (\text{composition})_{M,t_0} + \partial/\partial t\,(\text{composition})_M\,\delta t$$

- *Calculate modified structure for new composition and repeat* to discover how star evolves (not valid if stellar properties change so rapidly that time dependent terms in (1) to (4) cannot be ignored).

4.1.3 Convective regions: (GZ: 16-1; CO: 10.4)

- Equations (1) to (3) unchanged.
- for efficient convection (neutral buoyancy):

$$\frac{P}{T}\frac{dT}{dP} = \frac{\gamma - 1}{\gamma} \quad (4b)$$

- L_{rad} is calculated from equation (4) once the above have been solved.

4.2 THE EQUATION OF STATE

4.2.1 Perfect gas: (GZ: 16-1: CO: 10.2)

$$P = NkT = \frac{\rho}{\mu m_H} kT$$

N is the number density of particles; μ is the mean particle mass in units of m_H. Define:

X = mass fraction of hydrogen (Sun: 0.70)

Y = mass fraction of helium (Sun: 0.28)

Z = mass fraction of heavier elements (metals) (Sun: 0.02)

- $X + Y + Z = 1$
- If the material is assumed to be *fully ionized:*

Element	No. of atoms	No. of electrons
Hydrogen	$X\rho/m_H$	$X\rho/m_H$
Helium	$Y\rho/4m_H$	$2Y\rho/4m_H$
Metals	$[Z\rho/(Am_H)]$	$(1/2)AZ\rho/(Am_H)$

- A represents the average atomic weight of heavier elements; each metal atom contributes $\sim A/2$ electrons.
- Total number density of particles:

  $N = (2X + 3Y/4 + Z/2)\,\rho/m_H$

  $\triangleright\ (1/\mu) = 2\,X + 3/4\,Y + 1/2\,Z$

- This is a good approximation to μ *except in cool, outer regions.*
- When Z is negligible: $Y = 1 - X$; $\mu = 4/(3 + 5X)$
- Inclusion of *radiation pressure* in P:

$$P = \rho kT/(\mu m_H) + aT^4/3.$$

(important for massive stars)

4.2.2 Degenerate gas: (GZ: 17-1; CO: 15.3)

- First deviation from perfect gas law in stellar interior occurs when electrons become degenerate.
- The *number density of electrons* in phase space is *limited by the Pauli exclusion principle.*

$$n_e\, dp_x dp_y dp_z\, dxdydz \le (2/h^3)\, dp_x dp_y dp_z\, dxdydz$$

- In a *completely degenerate gas* all cells for momenta smaller than a threshold momentum p_0 are completely filled (Fermi momentum).
- The number density of electrons within a sphere of radius p_0 in momentum space is (at $T = 0$):

$$N_e = \int_0^{p_0} (2/h^3)\, 4\pi p^2 dp = (2/h^3)(4\pi/3)p_0^3$$

- From *kinetic theory*

$$P_e = (1/3) \int_0^\infty p\, v(p) n(p) dp$$

(a) Non-relativistic complete degeneracy:

$$v(p) = p/m_e \qquad \text{for all } p$$

$$P_e = (1/3) \int_0^{p_0} (p^2/m_e)(2/h^3)\, 4\pi p^2\, dp$$

$$= \{8\pi/(15 m_e h^3)\} p_0^5 = \{h^2/(20 m_e)\}(3/\pi)^{2/3} N_e^{5/3}.$$

(b) Relativistic complete degeneracy:

$$\mathbf{v(p) \sim c}$$

$$\mathbf{P_e = (1/3) \int_0^{p_0} pc(2/h^3)\, 4\pi p^2\, dp}$$

$$\mathbf{= (8\pi c/3h^3) p_0^4/4 = (2\pi c/3h^3)\, p_0^4}$$

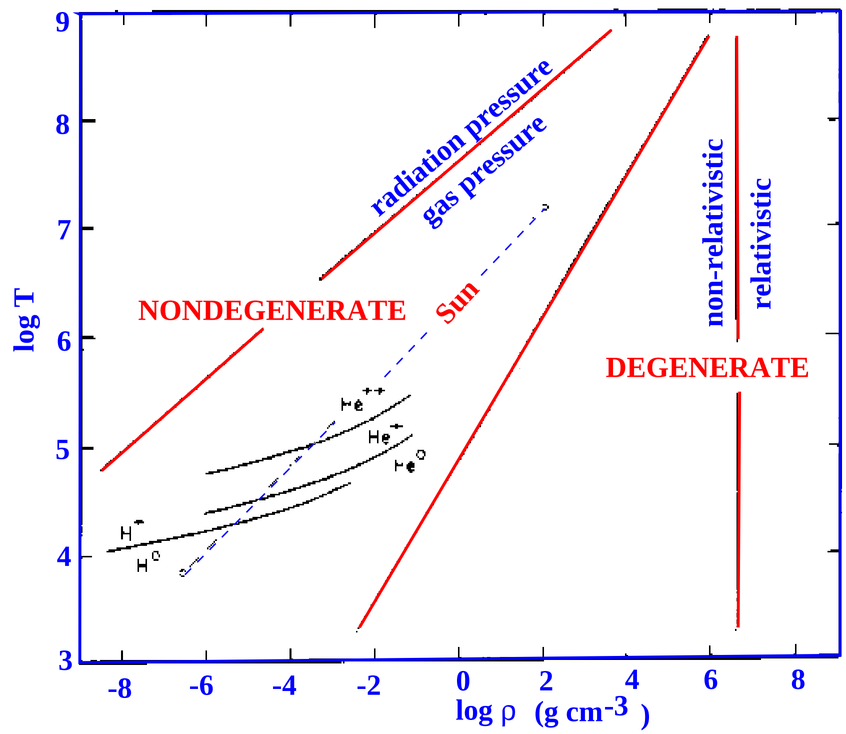
$$\mathbf{= (hc/8)(3/\pi)^{1/3}\, N_e^{4/3}.}$$

- **Also** $\mathbf{N_e = (X + Y/2 + Z/2)\, \rho/m_H = (1/2)(1 + X)\, \rho/m_H}$.
- **For intermediate regions use the full relativistic expression for v(p).**
- **For ions we may continue to use the non-degenerate equation:**
- $\mathbf{P_{ions} = (1/\mu_{ions})(\rho k T/m_H)}$ **where** $\mathbf{(1/\mu_{ions}) = X + Y/4}$.

Conditions where degeneracy is important:

(a) ***Non-relativistic*** **– interiors of** ***white dwarfs***; ***degenerate cores of red giants.***

(b) ***Relativistic*** **- very high densities only; interiors of** ***white dwarfs.***



Temperature-density diagram for the equation of state (Schwarzschild 1958)

4.3 THE OPACITY (GZ: 10-2; CO: 9.2)

The rate at which energy flows by radiative transfer is determined by the opacity *(cross section per unit mass [m^2/kg])*

$$dT/dr = -3\kappa L\rho/(16\pi acr^2T^3) \qquad (4)$$

In degenerate stars a similar equation applies with the opacity representing resistance to energy transfer by electron conduction.

Sources of stellar opacity:

1. *bound-bound absorption* (negligible in interiors)
2. *bound-free absorption*
3. *free-free absorption*
4. *scattering by free electrons*

- usually use a mean opacity averaged over frequency, *Rosseland mean opacity* (see textbooks).

Approximate analytical forms for opacity:

High temperature: $\kappa = \kappa_1 = 0.020\,\mathrm{m^2\,kg^{-1}}\,(1+X)$

Intermediate temperature: $\kappa = \kappa_2\rho T^{-3.5}$ *(Kramer's law)*

Low temperature: $\kappa = \kappa_3\rho^{1/2}T^4$

- $\kappa_1, \kappa_2, \kappa_3$ are constant for stars of given chemical composition but all depend on composition.

Opacities