6.1 THE STRUCTURE OF MAIN-SEQUENCE STARS (ZG: 16.2; CO 10.6, 13.1)

- main-sequence phase: *hydrogen core burning* phase
  - ▹ *zero-age main sequence (ZAMS): homogeneous* composition

Scaling relations for main-sequence stars

- use *dimensional analysis* to derive scaling relations (relations of the form $L \propto M^\gamma$)
- replace differential equations by characteristic quantities (e.g. $dP/dr \sim P/R$, $\rho \sim M/R^3$)
- *hydrostatic equilibrium* $\rightarrow P \sim \frac{GM^2}{R^4}$ (1)
- *radiative transfer* $\rightarrow L \propto \frac{R^4T^4}{\kappa M}$ (2)
- to derive *luminosity–mass relationship,* specify *equation of state* and *opacity law*

(1) *massive stars:* ideal-gas law, electron scattering opacity, i.e.

▹ $P = \frac{\rho}{\mu m_H} kT \sim \frac{kT}{\mu m_H}\left(\frac{M}{R^3}\right)$ and $\kappa \simeq \kappa_{Th} = \text{constant}$

$$\Rightarrow \frac{kT}{\mu m_H} \sim \frac{GM}{R} \quad (3)$$

▹ substituting (3) into (2): $L \propto \frac{\mu^4 M^3}{\kappa_{Th}}$

(2) *low-mass stars:* ideal-gas law, Kramer's opacity law, i.e. $\kappa \propto \rho T^{-3.5}$

$$\Rightarrow L \propto \frac{\mu^{7.5} M^{5.5}}{R^{0.5}}$$

- *mass–radius relationship*
  - ▹ central temperature determined by characteristic nuclear-burning temperature (hydrogen fusion: $T_c \sim 10^7$ K; helium fusion: $T_c \sim 10^8$ K)
  - ▹ from (3) $\Rightarrow R \propto M$ (in reality $R \propto M^{0.6-0.8}$)

(3) *very massive stars:* radiation pressure, electron scattering opacity, i.e.

▹ $P = \frac{1}{3}aT^4 \rightarrow T \sim \frac{M^{1/2}}{R} \Rightarrow L \propto M$

- power-law index in mass–luminosity relationship decreases from ~ 5 *(low-mass)* to 3 *(massive)* and 1 *(very massive)*
- near a solar mass: $L \simeq L_\odot \left(\frac{M}{M_\odot}\right)^4$
- *main-sequence lifetime:* $T_{MS} \propto M/L$

$$\text{typically: } T_{MS} = 10^{10}\,\text{yr}\left(\frac{M}{M_\odot}\right)^{-3}$$

- *pressure* is inverse proportional to the *mean molecular weight* μ
  - ▹ higher μ (fewer particles) implies higher temperature to produce the same pressure, but T_c *is fixed* (hydrogen burning *(thermostat):* $T_c \sim 10^7$ K)
  - ▹ during H-burning μ increases from ~ 0.62 to ~ 1.34

  $\rightarrow$ *radius increases* by a factor of ~ 2 (equation [3])

- *opacity* at low temperatures depends strongly on *metallicity* (for bound-free opacity: $\kappa \propto Z$)
  - ▷ *low-metallicity stars* are much *more luminous* at a given mass and have proportionately shorter lifetimes
  - ▷ mass-radius relationship only weakly dependent on metallicity
  - → low-metallicity stars are *much hotter*
  - ▷ *subdwarfs:* low-metallicity main-sequence stars lying just below the main sequence

General properties of homogeneous stars:

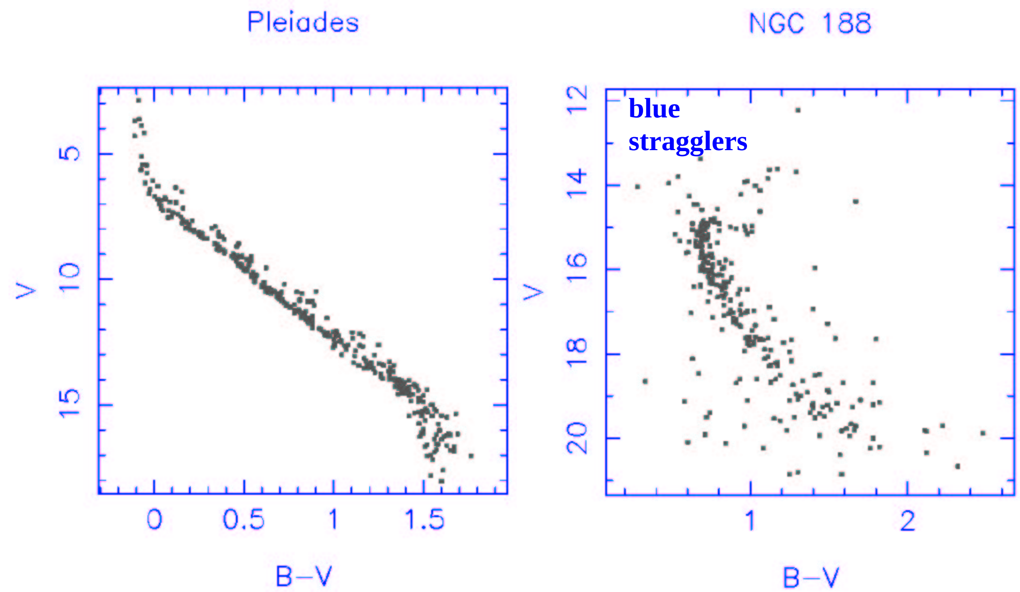
	Upper main sequence ($M_s > 1.5\,M_\odot$)	Lower main sequence ($M_s < 1.5\,M_\odot$)
core	*convective;* well mixed	*radiative*
ϵ	*CNO cycle*	*PP chain*
κ	*electron scattering*	*Kramer's opacity* $\kappa \simeq \kappa_3 \rho T^{-3.5}$
surface	*H fully ionized* energy transport by *radiation*	*H/He neutral* *convection* zone just below surface

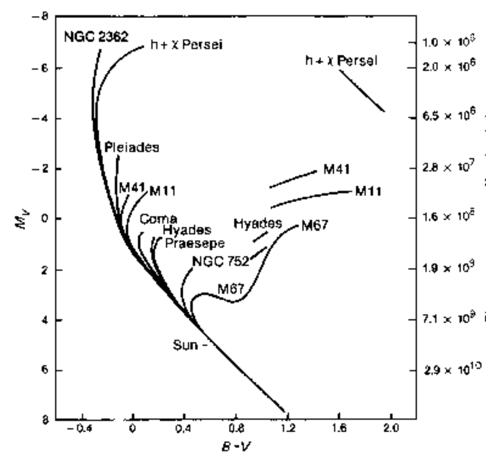
N.B. T_c increases with M_s; ρ_c decreases with M_s.

- *Hydrogen-burning limit:* $M_s \simeq 0.08\,M_\odot$
  - ▷ low-mass objects (brown dwarfs) do not burn hydrogen; they are supported by *electron degeneracy*
- maximum mass of stars: $100 - 150\,M_\odot$
- *Giants, supergiants and white dwarfs* cannot be chemically homogeneous stars supported by nuclear burning

Turnoff Ages in Open Clusters

$$T = 10^{10}\,\mathrm{yr}\left(\frac{L_{TO}}{L_\odot}\right)^{-3/4}$$





6.2 THE EVOLUTION OF LOW-MASS STARS
($M \lesssim 8\,M_\odot$) (ZG: 16.3; CO: 13.2)

6.2.1 Pre-main-sequence phase

- observationally new-born stars appear as *embedded protostars/T Tauri stars* near the *stellar birthline* where they burn *deuterium* ($T_c \sim 10^6\,K$), often still *accreting* from their birth clouds
- *after deuterium burning* → star *contracts*
  → $T_c \sim (\mu m_H/k)\,(GM/R)$ increases until hydrogen burning starts ($T_c \sim 10^7 K$) → main-sequence phase

6.2.2 Core hydrogen-burning phase

- energy source: *hydrogen burning* (4 H→ ^{4}He)
  → mean molecular weight μ increases in core from 0.6 to 1.3 → R, L and T_c increase (from $T_c \propto \mu\,(GM/R)$)
- lifetime: $T_{MS} \simeq 10^{10}\,yr \left(\frac{M}{M_\odot}\right)^{-3}$

after hydrogen exhaustion:

- formation of *isothermal core*
- *hydrogen burning in shell* around inert core (shell-burning phase)

→ growth of core until $M_{core}/M \sim 0.1$ *(Schönberg-Chandrasekhar limit)*

- ▷ core becomes too massive to be supported by thermal pressure

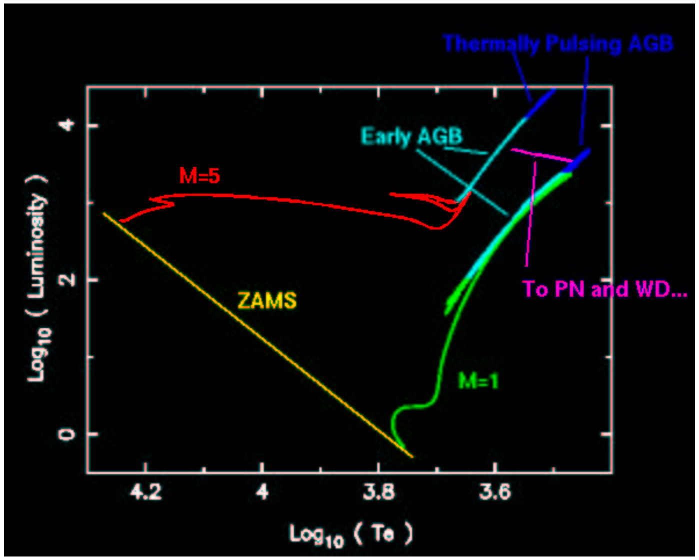
→ *core contraction* → energy source: *gravitational energy* → core becomes denser and hotter

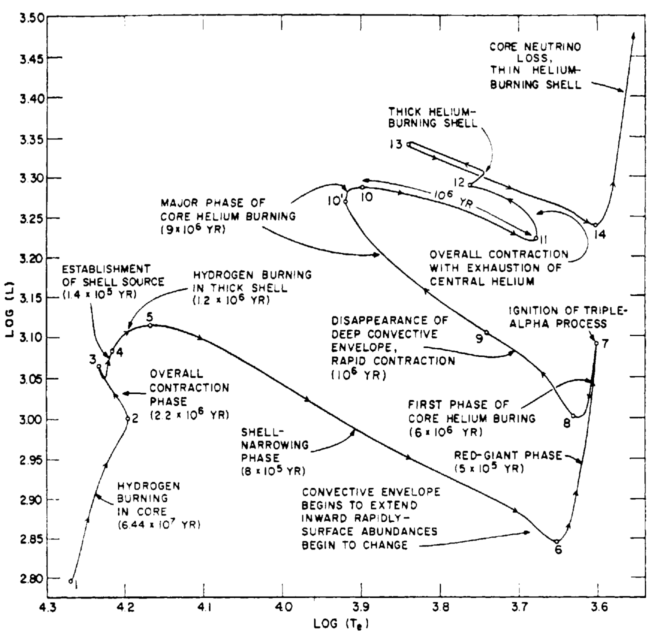
- ▷ contraction stops when the core density becomes high enough that *electron degeneracy pressure* can support the core

  (stars more massive than $\sim 2\,M_\odot$ ignite helium in the core before becoming degenerate)

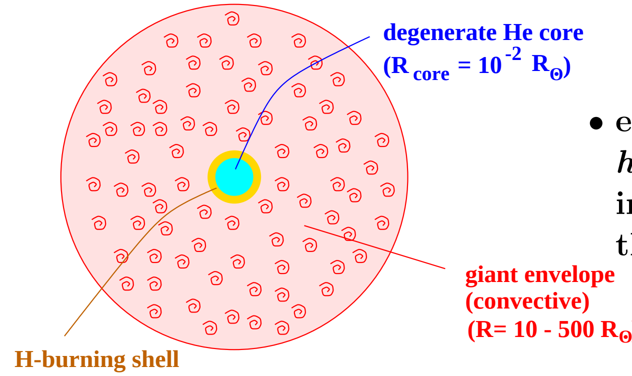
- while the core contracts and becomes degenerate, the *envelope expands* dramatically
  → star becomes a *red giant*
  - ▷ the transition to the red-giant branch is not well understood (in intuitive terms)
  - ▷ for stars with $M \gtrsim 1.5\,M_\odot$, the transition occurs very fast, i.e. on a thermal timescale of the envelope → few stars observed in transition region *(Hertzsprung gap)*

Evolutionary Tracks (1 to 5 $M_\odot$)





6.2.3 THE RED-GIANT PHASE

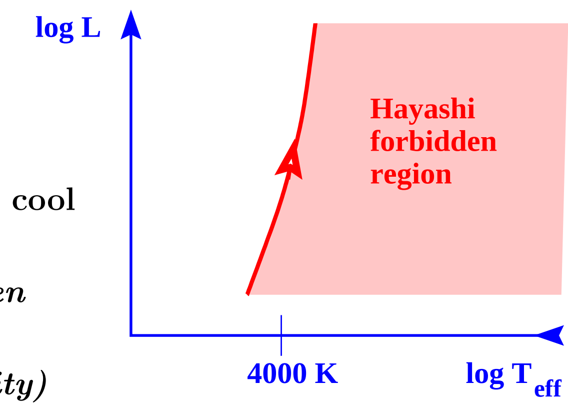


- energy source: *hydrogen burning* in thin *shell* above the degenerate core

- core mass grows → temperature in shell increases → luminosity increases → star *ascends red-giant branch*

- *Hayashi track:* vertical track in H-R diagram
  - ▷ no hydrostatic solutions for very cool giants
  - ▷ *Hayashi forbidden region*

    (due to H^- opacity)



- when the core mass reaches $M_c \simeq 0.48\,M_\odot$ → ignition of helium → *helium flash*

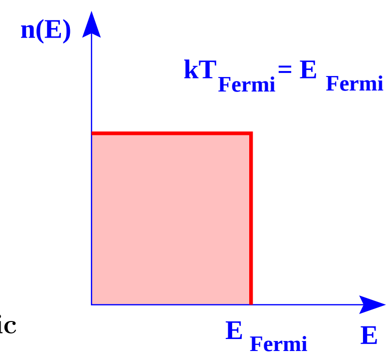
6.2.4 *HELIUM FLASH*

- *ignition of He under degenerate conditions* (for $M \lesssim 2\,M_\odot$; core mass $\sim 0.48\,M_\odot$)
  - ▷ i.e. P is independent of T → *no self-regulation*

    [in normal stars: increase in T → decrease in ρ (expansion) → decrease in T (virial theorem)]
  - ▷ in degenerate case: nuclear burning → increase in T → more nuclear burning → further increase in T

→ *thermonuclear runaway*

- runaway *stops* when matter becomes *non-degenerate* (i.e. $T \sim T_{\rm Fermi}$)
- disruption of star?
  - ▷ energy generated in runaway:
  - ▷ $\Delta E = \underbrace{\frac{M_{\rm burned}}{\mu m_H}}_{\text{number of particles}} \underbrace{kT_{\rm Fermi}}_{\text{characteristic energy}}$



$$\to \quad \Delta E \sim 2 \times 10^{42}\,{\rm J} \left(\frac{M_{\rm burned}}{0.1\,M_\odot}\right) \left(\frac{\rho}{10^9\,{\rm kg\,m^{-3}}}\right)^{2/3} \qquad (\mu \simeq 2)$$

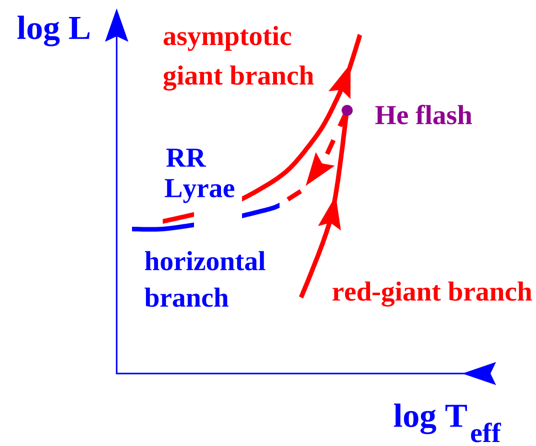
▷ compare ΔE to the binding energy of the core $E_{\rm bind} \simeq GM_c^2/R_c \sim 10^{43}\,{\rm J}$ ($M_c = 0.5\,M_\odot$; $R_c = 10^{-2}\,R_\odot$)

→ expect significant *dynamical expansion,* but no disruption ($t_{\rm dyn} \sim$ sec)

→ core expands → *weakening of H shell source* → rapid decrease in luminosity

→ star settles on *horizontal branch*

6.2.5 *THE HORIZONTAL BRANCH (HB)*

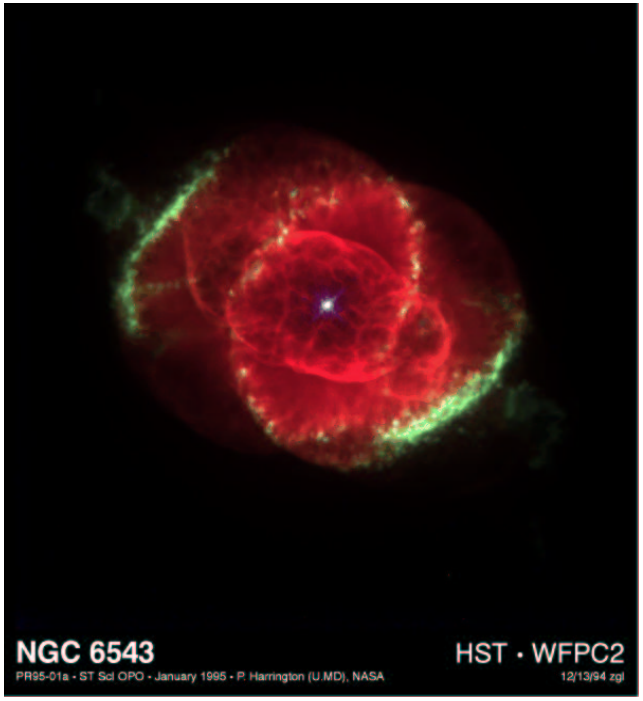


- *He burning* in center: conversion of He to mainly C and O ($^{12}C + \alpha \to ^{16}O$)
- *H burning* in shell (usually the dominant energy source)
- *lifetime:* $\sim 10\,\%$ of main-sequence lifetime (lower efficiency of He burning, higher luminosity)
- *RR Lyrae stars* are *pulsationally unstable* (L, B − V change with periods $\lesssim 1$ d)

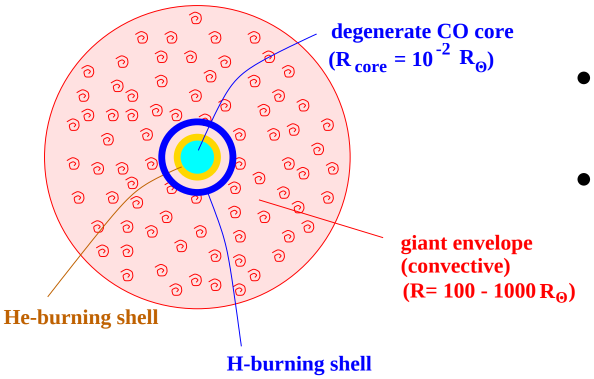
  easy to detect → popular *distance* indicators
- after *exhaustion of central He*

  → core *contraction* (as before) → *degenerate core* → *asymptotic giant branch*

Planetary Nebulae with the HST



6.2.6 THE ASYMPTOTIC GIANT BRANCH (AGB)



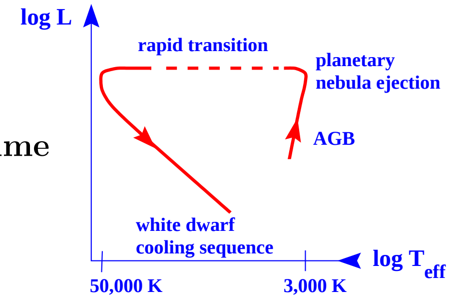
- *H burning* and *He burning* (in thin shells)
- H/He burning do not occur simultaneous, but alternate → *thermal pulsations*

- low-/intermediate-mass stars ($M \lesssim 8\,M_\odot$) do not experience nuclear burning beyond helium burning
- *evolution ends* when the *envelope* has been *lost* by stellar winds
  - ▷ *superwind phase:* very rapid mass loss ($\dot{M} \sim 10^{-4}\,M_\odot\,yr^{-1}$)
  - ▷ probably because envelope attains *positive binding energy* (due to energy reservoir in ionization energy)
  - → envelopes can be dispersed to infinity without requiring energy source
  - ▷ complication: radiative losses

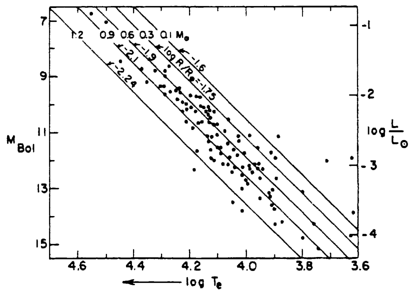
- after ejection:
  hot CO core is exposed
  and *ionizes* the ejected shell

→ *planetary nebula phase* (lifetime $\sim 10^4$ yr)

- CO core cools, becomes *degenerate* → *white dwarf*



6.2.7 WHITE DWARFS (ZG: 17.1; CO: 13.2)



	Mass ($M_\odot$)	Radius ($R_\odot$)
Sirius B	1.053 ± 0.028	0.0074 ± 0.0006
40 Eri B	0.48 ± 0.02	0.0124 ± 0.0005
Stein 2051	0.50 ± 0.05	0.0115 ± 0.0012

- first white dwarf discovered: *Sirius B* (companion of Sirius A)
  - ▷ mass (from orbit): $M \sim 1\,M_\odot$
  - ▷ radius (from $L = 4\pi R^2 \sigma T_{eff}^4$) $R \sim 10^{-2}\,R_\odot \sim R_\oplus$

$\rightarrow$ $\rho \sim 10^9\,\mathrm{kg\,m^{-3}}$

- Chandrasekhar (Cambridge 1930)
  - ▷ white dwarfs are supported by *electron degeneracy pressure*
  - ▷ white dwarfs have a *maximum mass* of $1.4\,M_\odot$
- most white dwarfs have a *mass* very close to $M \sim 0.6\,M_\odot$: $M_{WD} = 0.58 \pm 0.02\,M_\odot$
- most are made of carbon and oxygen *(CO white dwarfs)*
- some are made of He or O-Ne-Mg

Mass-Radius Relations for White Dwarfs

Non-relativistic degeneracy

- $P \sim P_e \propto (\rho/\mu_e m_H)^{5/3} \sim GM^2/R^4$

$$\rightarrow \boxed{R \propto \frac{1}{m_e} (\mu_e m_H)^{5/3} M^{-1/3}}$$

- note the *negative exponent*

$\rightarrow$ R decreases with increasing mass

$\rightarrow$ ρ increases with M

Relativistic degeneracy (when $p_{Fe} \sim m_e c$)

- $P \sim P_e \propto (\rho/\mu_e m_H)^{4/3} \sim GM^2/R^4$

$\rightarrow$ *M independent of R*

$\rightarrow$ existence of a *maximum mass*

THE CHANDRASEKHAR MASS

- consider a star of radius R containing N Fermions (electrons or neutrons) of mass m_f
- the mass per Fermion is $\mu_f m_H$ (μ_f = mean molecular weight per Fermion) $\rightarrow$ number density $n \sim N/R^3$ $\rightarrow$ volume/Fermion $1/n$
- ***Heisenberg uncertainty principle***
  $[\Delta x\, \Delta p \sim \hbar]^3 \rightarrow$ typical momentum: $p \sim \hbar\, n^{1/3}$

$\rightarrow$ ***Fermi energy*** of relativistic particle ($E = pc$)

$$E_f \sim \hbar\, n^{1/3}\, c \sim \frac{\hbar c\, N^{1/3}}{R}$$

- ***gravitational energy*** per Fermion

$$E_g \sim -\frac{GM(\mu_f m_H)}{R}, \text{ where } M = N\,\mu_f m_H$$

$\rightarrow$ total energy (per particle)

$$E = E_f + E_g = \frac{\hbar c N^{1/3}}{R} - \frac{GN(\mu_f m_H)^2}{R}$$

- stable configuration has minimum of total energy
- if $E < 0$, E can be decreased without bound by decreasing R $\rightarrow$ no equilibrium $\rightarrow$ *gravitational collapse*
- maximum N, if $E = 0$

$$\rightarrow\ N_{max} \sim \left(\frac{\hbar c}{G(\mu_f m_H)^2}\right)^{3/2} \sim 2 \times 10^{57}$$

$$M_{max} \sim N_{max}\,(\mu_e m_H) \sim 1.5\, M_\odot$$

Chandrasekhar mass for white dwarfs

$$\boxed{M_{Ch} = 1.457 \left(\frac{2}{\mu_e}\right)^2 M_\odot}$$

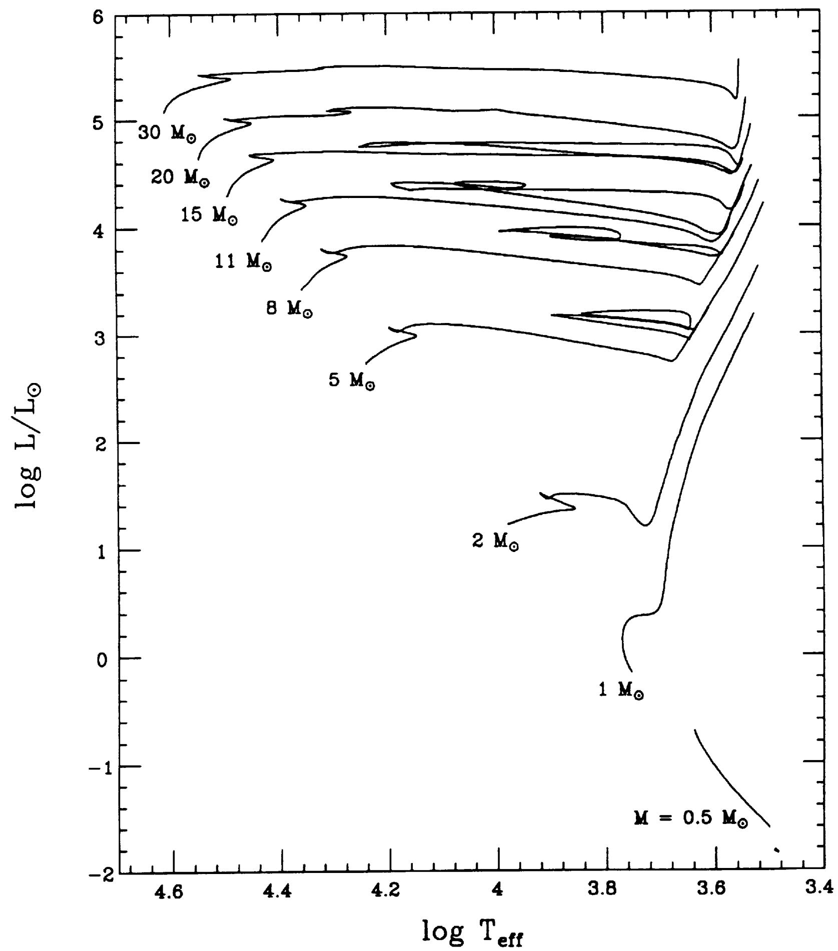


Figure B.1: Composite H-R diagram presenting the evolutionary tracks for stars between $0.5\,M_\odot$ and $30\,M_\odot$. The calculations assume an initially solar composition ($Y = 0.28$, $Z = 0.02$) and a mixing length parameter $\alpha = 1.5$.