## 6.1 THE STRUCTURE OF MAIN-SEQUENCE STARS
(ZG: 16.2; CO 10.6, 13.1)

- main-sequence phase: hydrogen core burning phase
  - ▷ zero-age main sequence (ZAMS): homogeneous composition

### Scaling relations for main-sequence stars

- use dimensional analysis to derive scaling relations (relations of the form $L \propto M^{\gamma}$)
- replace differential equations by characteristic quantities (e.g. $dP/dr \sim P/R$, $\rho \sim M/R^3$)
- hydrostatic equilibrium $\rightarrow P \sim \frac{GM^2}{R^4}$ (1)
- radiative transfer $\rightarrow L \propto \frac{R^4 T^4}{\kappa M}$ (2)
- to derive luminosity–mass relationship, specify equation of state and opacity law

(1) massive stars: ideal-gas law, electron scattering opacity, i.e.

▷ $P = \frac{\rho}{\mu m_H} kT \sim \frac{kT}{\mu m_H}\left(\frac{M}{R^3}\right)$ and $\kappa \simeq \kappa_{Th} = \text{constant}$

$$\Rightarrow \frac{kT}{\mu m_H} \sim \frac{GM}{R} \quad (3)$$

▷ substituting (3) into (2): $L \propto \frac{\mu^4 M^3}{\kappa_{Th}}$

(2) low-mass stars: ideal-gas law, Kramer's opacity law, i.e. $\kappa \propto \rho T^{-3.5}$

$$\Rightarrow L \propto \frac{\mu^{7.5} M^{5.5}}{R^{0.5}}$$

- mass–radius relationship
  - ▷ central temperature determined by characteristic nuclear-burning temperature (hydrogen fusion: $T_c \sim 10^7$ K; helium fusion: $T_c \sim 10^8$ K)
  - ▷ from (3) $\Rightarrow R \propto M$ (in reality $R \propto M^{0.6-0.8}$)

(3) very massive stars: radiation pressure, electron scattering opacity, i.e.

▷ $P = \frac{1}{3} a T^4 \rightarrow T \sim \frac{M^{1/2}}{R} \Rightarrow L \propto M$

- power-law index in mass–luminosity relationship decreases from $\sim 5$ (low-mass) to 3 (massive) and 1 (very massive)
- near a solar mass: $L \simeq L_\odot \left(\frac{M}{M_\odot}\right)^4$
- main-sequence lifetime: $T_{MS} \propto M/L$

  typically: $T_{MS} = 10^{10}\,\text{yr}\left(\frac{M}{M_\odot}\right)^{-3}$
- pressure is inverse proportional to the mean molecular weight $\mu$
  - ▷ higher $\mu$ (fewer particles) implies higher temperature to produce the same pressure, but $T_c$ is fixed (hydrogen burning (thermostat): $T_c \sim 10^7$ K)
  - ▷ during H-burning $\mu$ increases from $\sim 0.62$ to $\sim 1.34$

$\rightarrow$ radius increases by a factor of $\sim 2$ (equation [3])

- opacity at low temperatures depends strongly on metallicity (for bound-free opacity: $\kappa \propto Z$)
  - ▷ low-metallicity stars are much more luminous at a given mass and have proportionately shorter lifetimes
  - ▷ mass-radius relationship only weakly dependent on metallicity
  - → low-metallicity stars are much hotter
  - ▷ subdwarfs: low-metallicity main-sequence stars lying just below the main sequence

**General properties of homogeneous stars:**

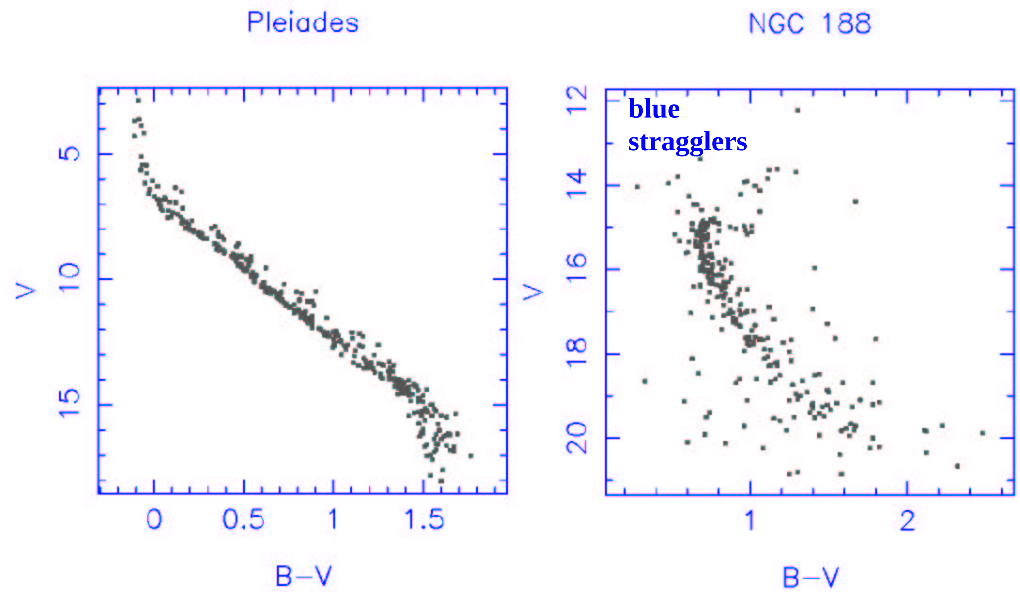
| | Upper main sequence ($M_s > 1.5\,M_\odot$) | Lower main sequence ($M_s < 1.5\,M_\odot$) |
|---|---|---|
| core | convective; well mixed | radiative |
| $\epsilon$ | CNO cycle | PP chain |
| $\kappa$ | electron scattering | Kramer's opacity $\kappa \simeq \kappa_3 \rho T^{-3.5}$ |
| surface | H fully ionized | H/He neutral |
| | energy transport by radiation | convection zone just below surface |

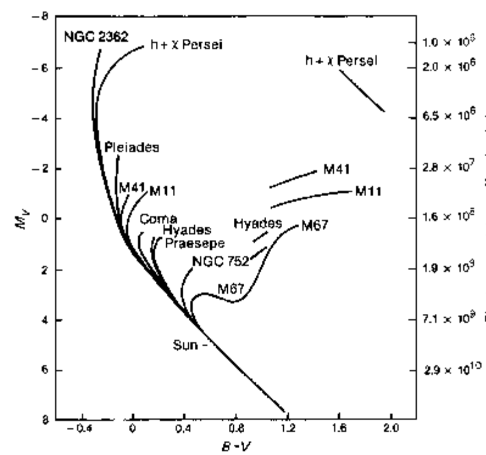
N.B. $T_c$ increases with $M_s$; $\rho_c$ decreases with $M_s$.

- Hydrogen-burning limit: $M_s \simeq 0.08\,M_\odot$
  - ▷ low-mass objects (brown dwarfs) do not burn hydrogen; they are supported by electron degeneracy
- maximum mass of stars: $100 - 150\,M_\odot$
- Giants, supergiants and white dwarfs cannot be chemically homogeneous stars supported by nuclear burning


## Turnoff Ages in Open Clusters

$$T = 10^{10}\,\text{yr} \left(\frac{L_{TO}}{L_\odot}\right)^{-3/4}$$



## 6.2 THE EVOLUTION OF LOW-MASS STARS ($M \lesssim 8\,M_\odot$) (ZG: 16.3; CO: 13.2)

### 6.2.1 Pre-main-sequence phase

- observationally new-born stars appear as embedded protostars/T Tauri stars near the stellar birthline where they burn deuterium ($T_c \sim 10^6\,K$), often still accreting from their birth clouds
- after deuterium burning → star contracts
  → $T_c \sim (\mu m_H/k)\,(GM/R)$ increases until hydrogen burning starts ($T_c \sim 10^7 K$) → main-sequence phase

### 6.2.2 Core hydrogen-burning phase

- energy source: hydrogen burning ($4\ H \rightarrow {}^4He$)
  → mean molecular weight $\mu$ increases in core from 0.6 to 1.3 → R, L and $T_c$ increase (from $T_c \propto \mu\,(GM/R)$)
- lifetime: $T_{MS} \simeq 10^{10}\,yr \left(\dfrac{M}{M_\odot}\right)^{-3}$

after hydrogen exhaustion:

- formation of isothermal core
- hydrogen burning in shell around inert core (shell-burning phase)

→ growth of core until $M_{core}/M \sim 0.1$ (Schönberg-Chandrasekhar limit)

- ▷ core becomes too massive to be supported by thermal pressure

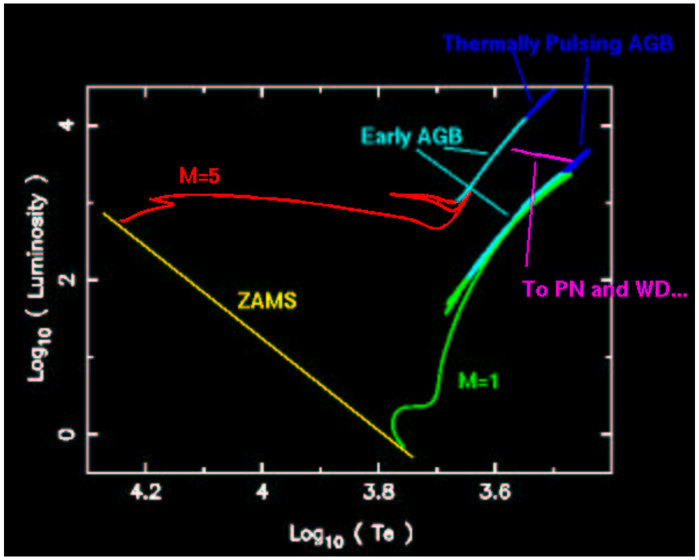
→ core contraction → energy source: gravitational energy → core becomes denser and hotter

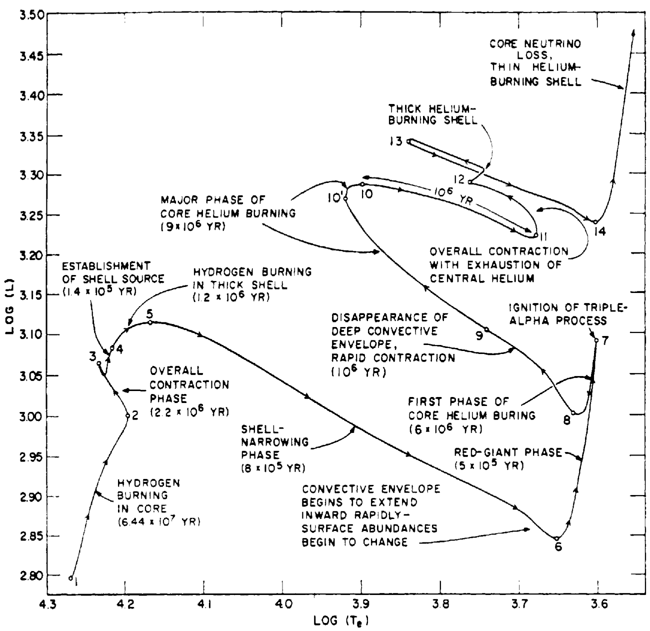
- ▷ contraction stops when the core density becomes high enough that electron degeneracy pressure can support the core

  (stars more massive than $\sim 2\,M_\odot$ ignite helium in the core before becoming degenerate)

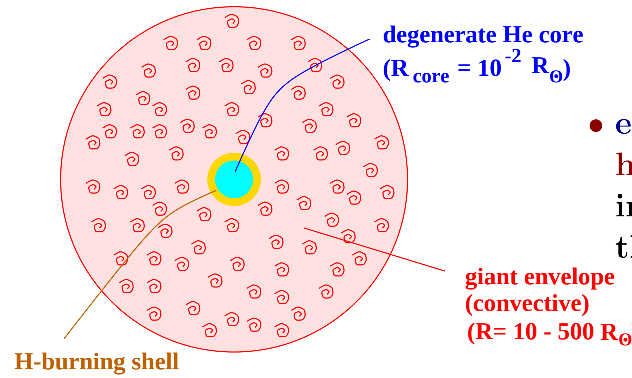
- while the core contracts and becomes degenerate, the envelope expands dramatically
  → star becomes a red giant
  - ▷ the transition to the red-giant branch is not well understood (in intuitive terms)
  - ▷ for stars with $M \gtrsim 1.5\,M_\odot$, the transition occurs very fast, i.e. on a thermal timescale of the envelope → few stars observed in transition region (Hertzsprung gap)

Evolutionary Tracks (1 to 5$M_\odot$)



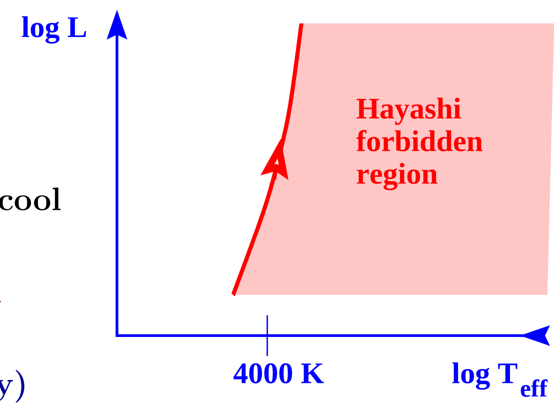




## 6.2.3 THE RED-GIANT PHASE




- energy source: hydrogen burning in thin shell above the degenerate core
- core mass grows $\rightarrow$ temperature in shell increases $\rightarrow$ luminosity increases $\rightarrow$ star ascends red-giant branch
- Hayashi track: vertical track in H-R diagram
  - ▷ no hydrostatic solutions for very cool giants
  - ▷ Hayashi forbidden region

  (due to $H^-$ opacity)




- when the core mass reaches $M_c \simeq 0.48\,M_\odot \rightarrow$ ignition of helium $\rightarrow$ helium flash

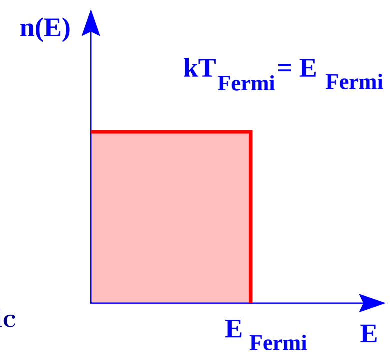
### 6.2.4 HELIUM FLASH

- ignition of He under degenerate conditions (for $M \lesssim 2\,M_\odot$; core mass $\sim 0.48\,M_\odot$)
  - ▷ i.e. P is independent of T → no self-regulation

    [in normal stars: increase in T → decrease in $\rho$ (expansion) → decrease in T (virial theorem)]
  - ▷ in degenerate case: nuclear burning → increase in T → more nuclear burning → further increase in T

→ thermonuclear runaway

- runaway stops when matter becomes non-degenerate (i.e. $T \sim T_{\rm Fermi}$)
- disruption of star?
  - ▷ energy generated in runaway:
  - ▷ $\Delta E = \underbrace{\frac{M_{\rm burned}}{\mu m_H}}_{\text{number of particles}} \underbrace{kT_{\rm Fermi}}_{\text{characteristic energy}}$




→ $\Delta E \sim 2 \times 10^{42}\,\mathrm{J} \left(\frac{M_{\rm burned}}{0.1\,M_\odot}\right) \left(\frac{\rho}{10^9\,\mathrm{kg\,m^{-3}}}\right)^{2/3} \qquad (\mu \simeq 2)$

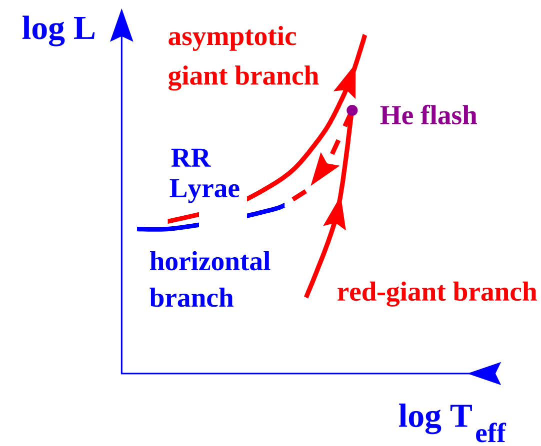
▷ compare $\Delta E$ to the binding energy of the core $E_{\rm bind} \simeq GM_c^2/R_c \sim 10^{43}\,\mathrm{J}$ ($M_c = 0.5\,M_\odot$; $R_c = 10^{-2}\,R_\odot$)

→ expect significant dynamical expansion, but no disruption ($t_{\rm dyn} \sim$ sec)

→ core expands → weakening of H shell source → rapid decrease in luminosity

→ star settles on horizontal branch

### 6.2.5 THE HORIZONTAL BRANCH (HB)

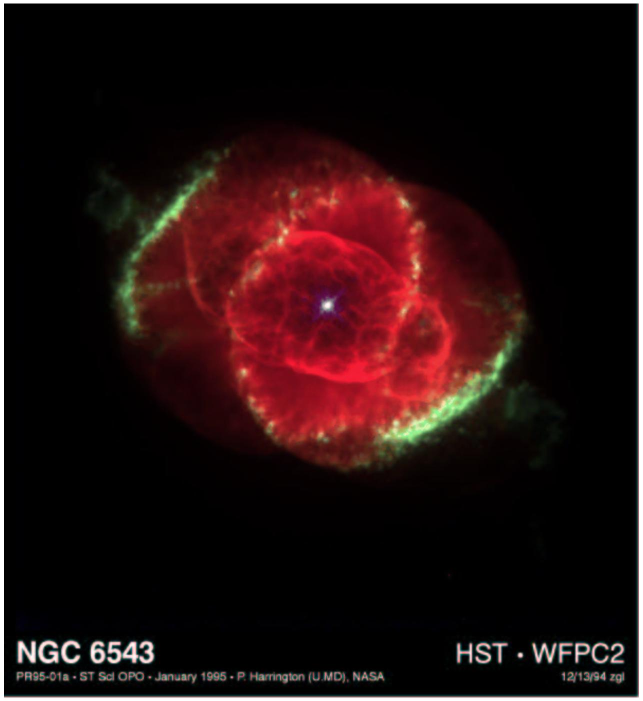



- He burning in center: conversion of He to mainly C and O ($^{12}C + \alpha \rightarrow ^{16}O$)
- H burning in shell (usually the dominant energy source)
- lifetime: $\sim 10\,\%$ of main-sequence lifetime (lower efficiency of He burning, higher luminosity)
- RR Lyrae stars are pulsationally unstable (L, B – V change with periods $\lesssim 1$ d)

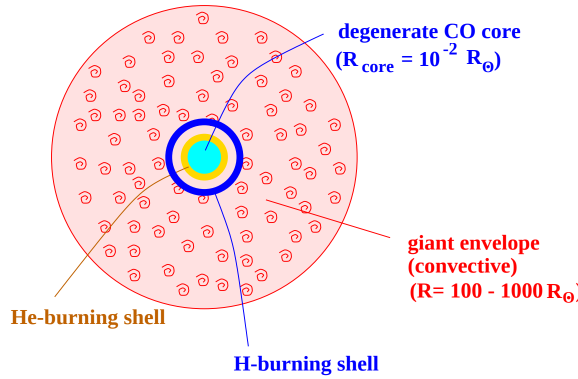
  easy to detect → popular distance indicators
- after exhaustion of central He

  → core contraction (as before) → degenerate core → asymptotic giant branch

**Planetary Nebulae with the HST**




## 6.2.6 THE ASYMPTOTIC GIANT BRANCH (AGB)




- H burning and He burning (in thin shells)
- H/He burning do not occur simultaneous, but alternate $\rightarrow$ thermal pulsations

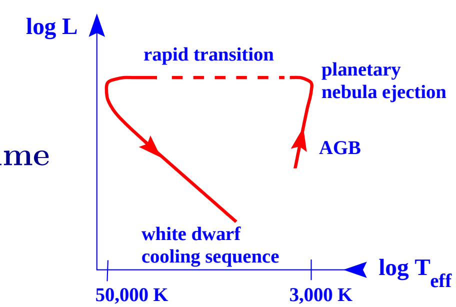
- low-/intermediate-mass stars $(M \lesssim 8\,M_\odot)$ do not experience nuclear burning beyond helium burning
- evolution ends when the envelope has been lost by stellar winds
  - ▷ superwind phase: very rapid mass loss $(\dot{M} \sim 10^{-4}\,M_\odot\,\mathrm{yr}^{-1})$
  - ▷ probably because envelope attains positive binding energy (due to energy reservoir in ionization energy)

  $\rightarrow$ envelopes can be dispersed to infinity without requiring energy source
  - ▷ complication: radiative losses

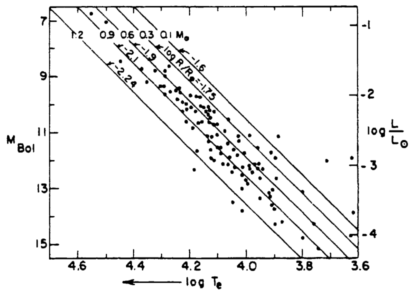
- after ejection:
  hot CO core is exposed
  and ionizes the ejected shell

$\rightarrow$ planetary nebula phase (lifetime $\sim 10^4\,\mathrm{yr}$)

- CO core cools, becomes degenerate $\rightarrow$ white dwarf



## 6.2.7 WHITE DWARFS (ZG: 17.1; CO: 13.2)




| | Mass ($M_\odot$) | Radius ($R_\odot$) |
|---|---|---|
| Sirius B | $1.053 \pm 0.028$ | $0.0074 \pm 0.0006$ |
| 40 Eri B | $0.48 \pm 0.02$ | $0.0124 \pm 0.0005$ |
| Stein 2051 | $0.50 \pm 0.05$ | $0.0115 \pm 0.0012$ |

- first white dwarf discovered: Sirius B (companion of Sirius A)
  - ▷ mass (from orbit): $M \sim 1\,M_\odot$
  - ▷ radius (from $L = 4\pi R^2 \sigma T_{\rm eff}^4$) $R \sim 10^{-2}\,R_\odot \sim R_\oplus$

$\rightarrow$ $\rho \sim 10^9\,\mathrm{kg\,m^{-3}}$

- Chandrasekhar (Cambridge 1930)
  - ▷ white dwarfs are supported by electron degeneracy pressure
  - ▷ white dwarfs have a maximum mass of $1.4\,M_\odot$
- most white dwarfs have a mass very close to $M \sim 0.6\,M_\odot$: $M_{\rm WD} = 0.58 \pm 0.02\,M_\odot$
- most are made of carbon and oxygen (CO white dwarfs)
- some are made of He or O-Ne-Mg

### Mass-Radius Relations for White Dwarfs

**Non-relativistic degeneracy**

- $P \sim P_e \propto (\rho/\mu_e m_H)^{5/3} \sim GM^2/R^4$

$$\rightarrow \boxed{R \propto \frac{1}{m_e} (\mu_e m_H)^{5/3}\, M^{-1/3}}$$

- note the negative exponent
  - $\rightarrow$ R decreases with increasing mass
  - $\rightarrow$ $\rho$ increases with M

**Relativistic degeneracy** (when $p_{\rm Fe} \sim m_e c$)

- $P \sim P_e \propto (\rho/\mu_e m_H)^{4/3} \sim GM^2/R^4$

$\rightarrow$ M independent of R

$\rightarrow$ existence of a maximum mass

## THE CHANDRASEKHAR MASS

- consider a star of radius $\mathrm{R}$ containing $\mathrm{N}$ Fermions (electrons or neutrons) of mass $\mathrm{m_f}$
- the mass per Fermion is $\mu_\mathrm{f} \mathrm{m_H}$ ($\mu_\mathrm{f}$ = mean molecular weight per Fermion) $\rightarrow$ number density $\mathrm{n} \sim \mathrm{N/R^3}$ $\rightarrow$ volume/Fermion $1/\mathrm{n}$
- Heisenberg uncertainty principle $[\Delta x\, \Delta p \sim \hbar]^3$ $\rightarrow$ typical momentum: $\mathrm{p} \sim \hbar\, \mathrm{n}^{1/3}$

$\rightarrow$ Fermi energy of relativistic particle $(\mathrm{E} = \mathrm{pc})$

$$\mathrm{E_f} \sim \hbar\, \mathrm{n}^{1/3}\, \mathrm{c} \sim \frac{\hbar \mathrm{c}\, \mathrm{N}^{1/3}}{\mathrm{R}}$$

- gravitational energy per Fermion

$$\mathrm{E_g} \sim -\frac{\mathrm{GM}(\mu_\mathrm{f} \mathrm{m_H})}{\mathrm{R}}, \text{ where } \mathrm{M} = \mathrm{N}\, \mu_\mathrm{f} \mathrm{m_H}$$

$\rightarrow$ total energy (per particle)

$$\mathrm{E} = \mathrm{E_f} + \mathrm{E_g} = \frac{\hbar \mathrm{c} \mathrm{N}^{1/3}}{\mathrm{R}} - \frac{\mathrm{GN}(\mu_\mathrm{f} \mathrm{m_H})^2}{\mathrm{R}}$$

- stable configuration has minimum of total energy
- if $\mathrm{E} < 0$, $\mathrm{E}$ can be decreased without bound by decreasing $\mathrm{R}$ $\rightarrow$ no equilibrium $\rightarrow$ gravitational collapse
- maximum $\mathrm{N}$, if $\mathrm{E} = 0$

$\rightarrow$ $$\mathrm{N_{max}} \sim \left(\frac{\hbar \mathrm{c}}{\mathrm{G}(\mu_\mathrm{f} \mathrm{m_H})^2}\right)^{3/2} \sim 2 \times 10^{57}$$

$$\mathrm{M_{max}} \sim \mathrm{N_{max}}\, (\mu_\mathrm{e} \mathrm{m_H}) \sim 1.5\, \mathrm{M_\odot}$$

Chandrasekhar mass for white dwarfs

$$\boxed{\mathrm{M_{Ch}} = 1.457 \left(\frac{2}{\mu_\mathrm{e}}\right)^2 \mathrm{M_\odot}}$$

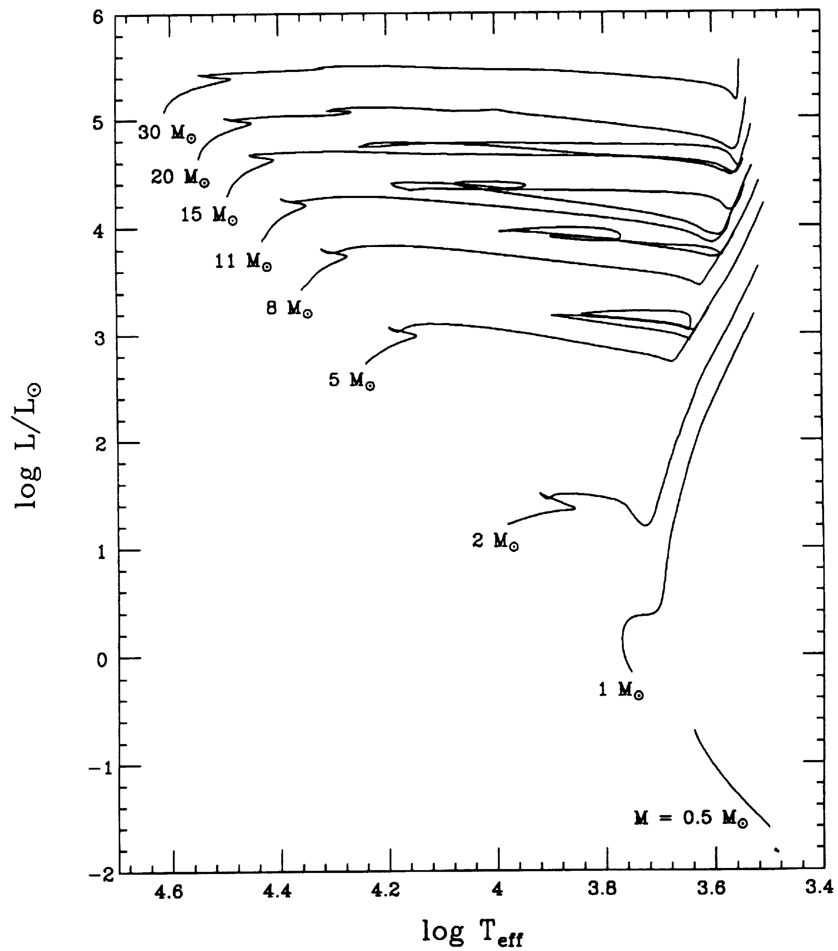



Figure B.1: Composite H-R diagram presenting the evolutionary tracks for stars between $0.5\, M_\odot$ and $30\, M_\odot$. The calculations assume an initially solar composition ($Y = 0.28$, $Z = 0.02$) and a mixing length parameter $\alpha = 1.5$.