

PHYSICAL STATE OF THE STELLAR INTERIOR

Fundamental assumptions:

- Although *stars evolve*, their properties change so *slowly* that at any time it is a good approximation to neglect the rate of change of these properties.
- Stars are *spherical* and symmetrical about their centres; all physical quantities depend just on r , the distance from the centre:

Equation of hydrostatic equilibrium:

*Fundamental principle 1: stars are self-gravitating bodies in dynamical equilibrium
→ balance of gravity and internal pressure forces*

Consider a small volume element at a distance r from the centre, cross section δS , length δr .

$$(P_{r+\delta r} - P_r) \delta S + GM_r (\rho_r \delta S \delta r) / r^2 = 0$$

$$\frac{dP_r}{dr} = -\frac{GM_r \rho_r}{r^2} \quad (1)$$

Equation of distribution of mass:

$$M_{r+\delta r} - M_r = (dM_r/dr) \delta r = 4\pi r^2 \rho_r \delta r$$

$$\frac{dM_r}{dr} = 4\pi r^2 \rho_r \quad (2)$$

Estimate of central pressure:

$$dP_r/dM_r = -GM_r/(4\pi r^4) \quad \text{from (1) and (2)}$$

$$-\int_0^{M_s} (dP_r/dM_r) dM_r = P_c - P_s = \int_0^{M_s} (GM_r/4\pi r^4) dM_r$$

At all points within the star $r < R_s$; hence $(1/r^4) > (1/R_s^4)$

$$\int_0^{M_s} (GM_r/4\pi r^4) dM_r > \int_0^{M_s} (GM_r/4\pi R_s^4) dM_r = (GM_s^2/8\pi R_s^4)$$

$$P_c > P_s + (GM_s^2/8\pi R_s^4) > (GM_s^2/8\pi R_s^4)$$

For the *Sun* $(P_c)_\odot > 4.5 \times 10^{13} \text{ N m}^{-2}$ or $4.5 \times 10^8 \text{ atm}$.

Crude estimate: dimensional analysis

– consider a point at $r = R_s/2$

$$dP_r/dr \sim P_c/R_s \quad \rho_r \sim \bar{\rho} = 3M_s/(4\pi R_s^3)$$

$$M_r \sim M_s/2 \quad P_c \sim (3/2\pi)(GM_s^2/R_s^4)$$

$$(P_c)_\odot \sim 5 \times 10^{14} \text{ N m}^{-2} \quad \text{or} \quad 5 \times 10^9 \text{ atm}$$

Estimate of central temperature:

Assume stellar material obeys the ideal gas equation

$$P_r = \frac{\rho_r}{\mu m_H} k T_r$$

μ = mean molecular weight in proton masses; $\mu \sim 1/2$ for fully ionized hydrogen.

$$T_c \sim \mu m_H P_c / (k \bar{\rho}), \quad m_H = 1.67 \times 10^{-27} \text{ kg}, \quad k = 1.4 \times 10^{-23} \text{ J K}^{-1}$$

$$(T_c)_\odot \sim 2 \times 10^7 \text{ K} \quad \bar{\rho}_\odot \sim 1.4 \times 10^3 \text{ kg m}^{-3} \quad (\text{c.f. } (T_s)_\odot \sim 5800 \text{ K})$$

- Although the Sun has a *mean density similar* to that of *water*, the high temperature requires that it should be *gaseous throughout*.
- the average *kinetic energy* of the particles is *higher* than the *binding energy of atomic hydrogen* so the material will be *highly ionized*, i.e is a *plasma*.

Dynamic timescale: t_D

- *Time for star to collapse* completely if pressure forces were negligible

$$(\rho \delta S \delta r) \ddot{r} = -(GM_r/r^2) (\rho \delta S \delta r)$$

- Inward displacement of element after time t is given by

$$s = (1/2) g t^2 = (1/2) (GM_r/r^2) t^2$$

- For estimate of t_D , put $s \sim R_s$, $r \sim R_s$, $M_r \sim M_s$; hence

$$t_D \sim (2R_s^3/GM_s)^{1/2} \sim \{3/(2\pi G\bar{\rho})\}^{1/2}$$

$$(t_D)_\odot \sim 2300 \text{ s} \sim 40 \text{ mins}$$

Stars adjust very quickly to maintain a balance between pressure and gravitational forces.

The virial theorem

$$dP_r/dr = -GM_r\rho_r/r^2$$

$$4\pi r^3 dP_r = -(GM_r/r) 4\pi r^2 \rho_r dr$$

$$4\pi [r^3 P_r]_{r=0, P=P_c}^{r=R_s, P=P_s} - 3 \int_0^{R_s} P_r 4\pi r^2 dr = - \int_0^{R_s} (GM_r/r) 4\pi r^2 \rho_r dr$$

$$\int_0^{R_s} 3P_r 4\pi r^2 dr = \int_0^{R_s} (GM_r/r) 4\pi r^2 \rho_r dr$$

Thermal energy/unit volume $u = nkT/2 = (\rho/\mu m_H)fkT/2$

Ratio of specific heats $\gamma = c_p/c_v = (f+2)/f$ ($f=3$: $\gamma=5/3$)

$$u = \{1/(\gamma-1)\}(\rho kT/\mu m_H) = P/(\gamma-1)$$

$$\boxed{3(\gamma-1)U + \Omega = 0}$$

U = total *thermal energy*; Ω = total *gravitational energy*.

For a fully ionized, ideal gas $\gamma=5/3$ and $2U + \Omega = 0$

Total energy of star $E = U + \Omega$

$$\boxed{E = -U = \Omega/2}$$

Note: E is negative and equal to $\Omega/2$ or $-U$. A *decrease in E* leads to a decrease in Ω but an *increase in U and hence T* . A star, with no hidden energy sources, composed of a perfect gas *contracts and heats up* as it radiates energy.

Fundamental principle 2: stars have a negative 'heat capacity', they heat up when their total energy decreases

Sources of stellar energy:

Fundamental principle 3: since stars lose energy by radiation, stars supported by thermal pressure require an energy source to avoid collapse

Provided stellar material always behaves as a perfect gas, thermal energy of star \sim gravitational energy.

- *total energy* available $\sim GM_s^2/2R_s$
- *thermal time-scale* (Kelvin-Helmholtz timescale):
 $t_{th} \sim GM_s^2/(2R_s L_s)$
- e.g. *Sun* radiates $L_\odot \sim 4 \times 10^{26}$ W and from geological evidence L_\odot has not changed significantly over $t \sim 10^9$ years
 $(t_{th})_\odot \sim GM_\odot^2/(R_\odot L_\odot) \sim 0.5 \times 10^{15}$ sec $\sim 1.5 \times 10^7$ years.

The thermal and gravitational energies of the Sun are not sufficient to cover radiative losses for the total solar lifetime.

Only nuclear energy can account for the observed luminosities and lifetimes of stars

- *Largest possible mass defect* available when H is transmuted into Fe: energy released = $0.008 \times$ total mass. For the Sun

$$(E_N)_\odot = 0.008 M_\odot c^2 \sim 10^{45} \text{ J}$$

- *Nuclear timescale* $(t_N)_\odot \sim (E_N)_\odot / L_\odot \sim 10^{11}$ yr

- *Energy loss at stellar surface* as measured by the stellar luminosity is *compensated by* energy release from *nuclear reactions* throughout the stellar interior.

$$L_s = \int_0^{R_s} \epsilon_r \rho_r \cdot 4\pi r^2 dr$$

ϵ_r is the nuclear energy released per unit mass per sec and will depend on T_r , ρ_r and composition.

$$\frac{dL_r}{dr} = 4\pi r^2 \rho_r \epsilon_r \quad (3)$$

for any elementary shell.

- During *rapid evolutionary phases*, (i.e. $t \ll t_{th}$)

$$\frac{dL_r}{dr} = 4\pi r^2 \rho_r \left(\epsilon_r - T \frac{dS}{dt} \right) \quad (3a),$$

where $-TdS/dt$ is referred to as a *gravitational energy* term.

Energy transport

The size of the *energy flux* is determined by the mechanism that provides the *energy transport: conduction, convection or radiation*. For all these mechanisms the *temperature gradient determines the flux*.

- *Conduction* does not contribute seriously to energy transport through the interior
 - ▷ At high gas density, mean free path for particles \ll mean free path for photons.
 - ▷ Special case, *degenerate matter* – very effective conduction by electrons.
- The *thermal radiation field* in the interior of a star consists mainly of *X-ray photons in thermal equilibrium with particles*.
- Stellar material is *opaque to X-rays* (bound-free absorption by inner electrons)
- *mean free path for X-rays* in solar interior ~ 1 cm.
- Photons reach the surface by a “*random walk*” process and as a result of many interactions with matter are degraded from X-ray to optical frequencies.
- After N steps of size l , the distribution has spread to $\simeq \sqrt{N}l$. For a photon to “random walk” a distance R_s , requires a *diffusion time* (in steps of size l)

$$t_{\text{diff}} = N \times \frac{l}{c} \simeq \frac{R_s^2}{lc}$$

For $l = 1$ cm, $R_s \sim R_\odot \rightarrow t_{\text{diff}} \sim 5 \times 10^3$ yr.

Energy transport by radiation:

- Consider a spherical shell of area $A = 4\pi r^2$, at radius r of thickness dr .
- *radiation pressure*

$$P_{\text{rad}} = \frac{1}{3}aT^4 \quad (i)$$

(=momentum flux)

- *rate of deposition of momentum* in region $r \rightarrow r + dr$

$$-\frac{dP_{\text{rad}}}{dr} dr 4\pi r^2 \quad (ii)$$

- define *opacity* κ [m^2/kg], so that fractional intensity loss in a beam of radiation is given by

$$\frac{dI}{I} = -\kappa \rho dx,$$

where ρ is the mass density and

$$\tau \equiv \int \kappa \rho dx$$

is called *optical depth* (note: $I = I_0 \exp(-\tau)$)

- ▷ $1/\kappa\rho$: mean free path
- ▷ $\tau \gg 1$: optically thick
- ▷ $\tau \ll 1$: optically thin
- rate of momentum absorption in shell $L(r)/c \kappa \rho dr$. Equating this with equation (ii) and using (i):

$$L_r = -4\pi r^2 \frac{4ac}{3\kappa\rho} T^3 \frac{dT}{dr} \quad (4a)$$

Energy transport by convection:

- *Convection* occurs in liquids and gases when the temperature gradient exceeds some typical value.
- *Criterion for stability against convection (Schwarzschild criterion)*
 - ▷ consider a *bubble* with initial $\bar{\rho}_1, \bar{P}_1$ *rising* by an amount dr , where the ambient pressure and density are given by $\rho(r), P(r)$.
 - ▷ the *bubble expands adiabatically*, i.e

$$\bar{P}_2 = \bar{P}_1 \left(\frac{\bar{\rho}_2}{\bar{\rho}_1} \right)^\gamma \quad (\gamma = \text{adiabatic exponent})$$

- ▷ assuming the bubble remains in *pressure equilibrium* with the ambient medium, i.e. $\bar{P}_2 = P_2 = P(r + dr) \simeq P_1 + (dP/dr) dr$,

$$\begin{aligned} \bar{\rho}_2 &= \bar{\rho}_1 \left(\frac{\bar{P}_2}{\bar{P}_1} \right)^{1/\gamma} \simeq \bar{\rho}_1 \left(1 + \frac{1}{P} \frac{dP}{dr} dr \right)^{1/\gamma} \\ &\simeq \bar{\rho}_1 + \frac{\bar{\rho}_1}{\gamma P} \frac{dP}{dr} dr \end{aligned}$$

- ▷ *convective stability* if $\bar{\rho}_2 - \rho_2 > 0$ (bubble will sink back)

$$\boxed{\frac{\rho}{\gamma P} \frac{dP}{dr} - \frac{d\rho}{dr} > 0}$$

- For a perfect gas (negligible radiation pressure)
$$P = \rho kT / (\mu m_H)$$
- Provided μ does not vary with position (no changes in ionization or dissociation)

$$-[1 - (1/\gamma)](T/P) dP/dr > -dT/dr \quad (\text{both negative})$$

- or *magnitude of adiabatic* $dT/dr > \text{magnitude of actual } dT/dr$.
- Alternatively, $\frac{P}{T} \frac{dT}{dP} < \frac{\gamma - 1}{\gamma}$
- There is *no generally accepted theory of convective energy transport* at present. The stability criterion must be checked at every layer within a stellar model: dP/dr from equation (1) and dT/dr from equation (4). The stability criterion can be broken in two ways:
 1. *Very high fluxes or very high opacities* can lead to *high (unstable) temperature gradients* e.g. in stellar cores.
 2. $(\gamma - 1)$ can be much smaller than $2/3$ for a monatomic gas, e.g. in *hydrogen ionization zones*.

Influence of convection

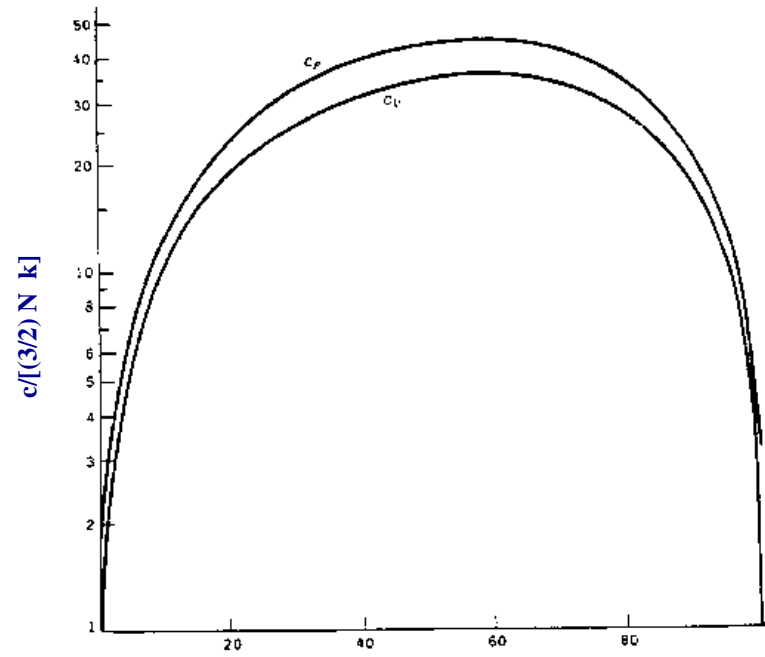
- (a) Motions are *turbulent*: too slow to disturb hydrostatic equilibrium.
- (b) *Highly efficient energy transport*: high thermal energy content of particles in stellar interior.
- (c) Turbulent *mixing* so *fast* that *composition* of convective region *homogeneous* at all times.
- (d) Actual dT/dr only exceeds adiabatic dT/dr by very slight amount.

Hence to sufficient accuracy (in convective regions)

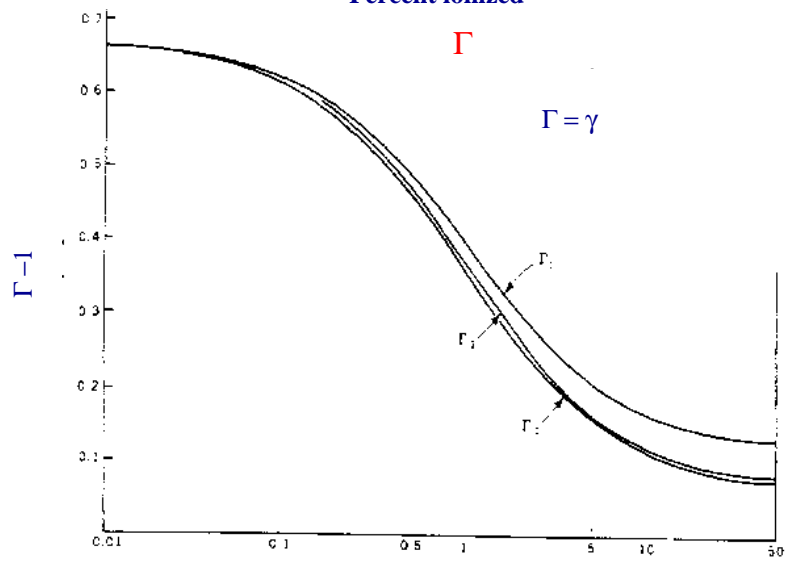
$$\boxed{\frac{dT}{dr} = \frac{\gamma - 1}{\gamma} \frac{T}{P} \frac{dP}{dr}} \quad (4b)$$

This is not a good approximation close to the surface (in particular for giants) where the density changes rapidly.

Specific Heats



Percent ionized



Percent ionized