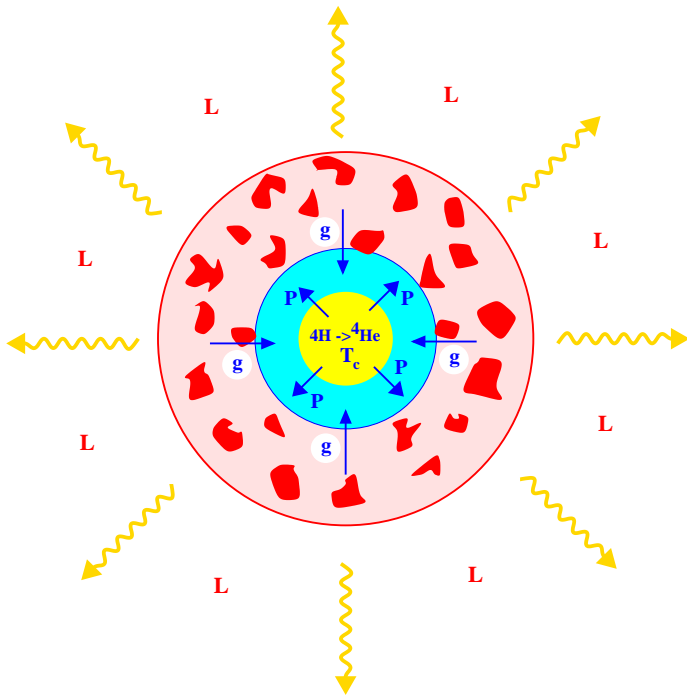


FUNDAMENTAL PRINCIPLES

- Stars are *self-gravitating* bodies in *dynamical equilibrium* → *balance of gravity and internal pressure forces* (hydrostatic equilibrium);
- stars lose energy by *radiation from the surface* → stars supported by thermal pressure require an *energy source* to avoid collapse, e.g. *nuclear energy, gravitational energy* (energy equation);
- the *temperature structure* is largely determined by the mechanisms by which *energy is transported* from the core to the surface, *radiation, convection, conduction* (energy transport equation);
- the *central temperature* is determined by the *characteristic temperature* for the appropriate *nuclear fusion reactions* (e.g. H-burning: 10^7 K; He-burning: 10^8 K);
- normal stars have a *negative ‘heat capacity’* (virial theorem): they heat up when their total energy decreases (→ normal stars contract and heat up when there is no nuclear energy source);
- *nuclear burning is self-regulating* in non-degenerate cores (virial theorem): e.g. a sudden increase in nuclear burning causes expansion and cooling of the core: *negative feedback* → *stable nuclear burning*;
- the *global structure* of a star is determined by the *simultaneous satisfaction* of these principles → the *local properties* of a star are determined by the *global structure*.
(Mathematically: it requires the simultaneous solution of a set of coupled, non-linear differential equations with mixed boundary conditions.)



THE EQUATIONS OF STELLAR STRUCTURE

In the absence of convection:

$$\begin{aligned} \frac{dP_r}{dr} &= \frac{-GM_r\rho_r}{r^2} & (1) \\ \frac{dM_r}{dr} &= 4\pi r^2 \rho_r & (2) \\ \frac{dL_r}{dr} &= 4\pi r^2 \rho_r \left(\varepsilon_r - T \frac{dS}{dt} \right) & (3) \\ \frac{dT_r}{dr} &= \frac{-3\kappa_r L_r \rho_r}{16\pi a c r^2 T_r^3} & (4a) \end{aligned}$$

- $P_r, \kappa_r, \varepsilon_r$ are functions of ρ, T , chemical composition
- Basic physics can provide expressions for these.
- In total, there are *four, coupled, non-linear, partial differential equations (+ three constitutive relations) for seven unknowns*: $P, \rho, T, M, L, \kappa, \varepsilon$ as functions of r .
- These completely determine the structure of a star of given composition subject to suitable boundary conditions.
- In general, only numerical solutions can be obtained (i.e. computer).
- *Four (mixed) boundary conditions* needed:
 - ▷ at centre: $M_r = 0$ and $L_r = 0$ at $r = 0$ (exact)
 - ▷ at surface: $L_s = 4\pi R_s^2 \sigma T_{\text{eff}}^4$ (blackbody relation)
(surface = photosphere, where $\tau \simeq 1$)
 $P = (2/3) g/\kappa$ (atmosphere model)
(sometimes: $P(R_s) = 0$ [rough], but *not* $T(R_s) = 0$)

- Use of *mass as independent variable* (more convenient from a theoretical point of view):

$$dP/dM = -GM/(4\pi r^4) \quad (1)$$

$$dr/dM = 1/(4\pi r^2 \rho) \quad (2)$$

$$dL/dM = \varepsilon - T dS/dt \quad (3)$$

$$dT/dM = -3\kappa L/(64\pi^2 a c r^4 T^3) \quad (4a)$$

- Boundary conditions: $r = 0, L = 0$, at $M = 0$;
 $L_s = 4\pi R_s^2 \sigma T_{\text{eff}}^4$ and $P = (2/3) g/\kappa$ at $M = M_s$.
- These equations must be solved for specified M_s and composition.

Uniqueness of solution: the Vogt Russell “Theorem”

“For a given chemical composition, only a single equilibrium configuration exists for each mass; thus the internal structure of the star is fixed.”

- This “theorem” has not been proven and is not even rigorously true; there are known exceptions (for special cases)

The equilibrium solution and stellar evolution:

- If there is *no bulk motions* in the interior of a star (i.e. no convection), *changes of chemical composition* are *localised in regions of nuclear burning* The structure equations (1) to (4) can be supplemented by equations of the type:

$$\partial/\partial t (\text{composition})_M = f(\rho, T, \text{composition})$$

Knowing the composition as a function of M at a time t_0 we can solve (1) to (4) for the structure at t_0 . Then

$$(\text{composition})_{M,t_0+\delta t} = (\text{composition})_{M,t_0} + \frac{\partial}{\partial t} (\text{composition})_M \delta t$$

- *Calculate modified structure for new composition and repeat* to discover how star evolves (not valid if stellar properties change so rapidly that time dependent terms in (1) to (4) cannot be ignored).

Structure equations in a convective core:

- Equations (1) to (3) unchanged.
- for efficient convection (neutral buoyancy):

$$\frac{P dT}{T dP} = \frac{\gamma - 1}{\gamma} \quad (4b)$$

- L_{rad} is calculated from equation (4) once the above have been solved.

THE EQUATION OF STATE

Perfect gas:

$$P = NkT = \frac{\rho}{\mu m_H} kT$$

N is the number density of particles; μ is the mean particle mass in units of m_H . Define:

X = mass fraction of hydrogen (Sun: 0.70)

Y = mass fraction of helium (Sun: 0.28)

Z = mass fraction of heavier elements (metals) (Sun: 0.02)

- $X + Y + Z = 1$
- If the material is assumed to be *fully ionized*:

Element	No. of atoms	No. of electrons
Hydrogen	$X\rho/m_H$	$X\rho/m_H$
Helium	$Y\rho/4m_H$	$2Y\rho/4m_H$
Metals	$[Z\rho/(A m_H)]$	$(1/2)AZ\rho/(A m_H)$

- A represents the average atomic weight of heavier elements; each metal atom contributes $\sim A/2$ electrons.
- Total number density of particles:
$$N = (2X + 3Y/4 + Z/2) \rho/m_H$$

$$\triangleright (1/\mu) = 2X + 3/4 Y + 1/2 Z$$
- This is a good approximation to μ *except in cool, outer regions*.

- When Z is negligible: $Y = 1 - X$; $\mu = 4/(3 + 5X)$
- Inclusion of *radiation pressure* in P :

$$P = \rho kT/(\mu m_H) + aT^4/3.$$

(important for massive stars)

Degenerate gas:

- First deviation from perfect gas law in stellar interior occurs when electrons become degenerate.
- The *number density of electrons* in phase space is *limited by the Pauli exclusion principle*.

$$n_e dp_x dp_y dp_z dx dy dz \leq (2/h^3) dp_x dp_y dp_z dx dy dz$$

- In a *completely degenerate gas* all cells for momenta smaller than a threshold momentum p_0 are completely filled (Fermi momentum).
- The number density of electrons within a sphere of radius p_0 in momentum space is (at $T = 0$):

$$N_e = \int_0^{p_0} (2/h^3) 4\pi p^2 dp = (2/h^3)(4\pi/3)p_0^3$$

- From *kinetic theory*

$$P_e = (1/3) \int_0^\infty p v(p) n(p) dp$$

(a) *Non-relativistic complete degeneracy:*

$$v(p) = p/m_e \quad \text{for all } p$$

$$P_e = (1/3) \int_0^{p_0} (p^2/m_e)(2/h^3) 4\pi p^2 dp$$

$$= \{8\pi/(15m_e h^3)\} p_0^5 = \{h^2/(20m_e)\} (3/\pi)^{2/3} N_e^{5/3}.$$

(b) *Relativistic complete degeneracy:*

$$v(p) \sim c$$

$$P_e = (1/3) \int_0^{p_0} pc(2/h^3) 4\pi p^2 dp$$

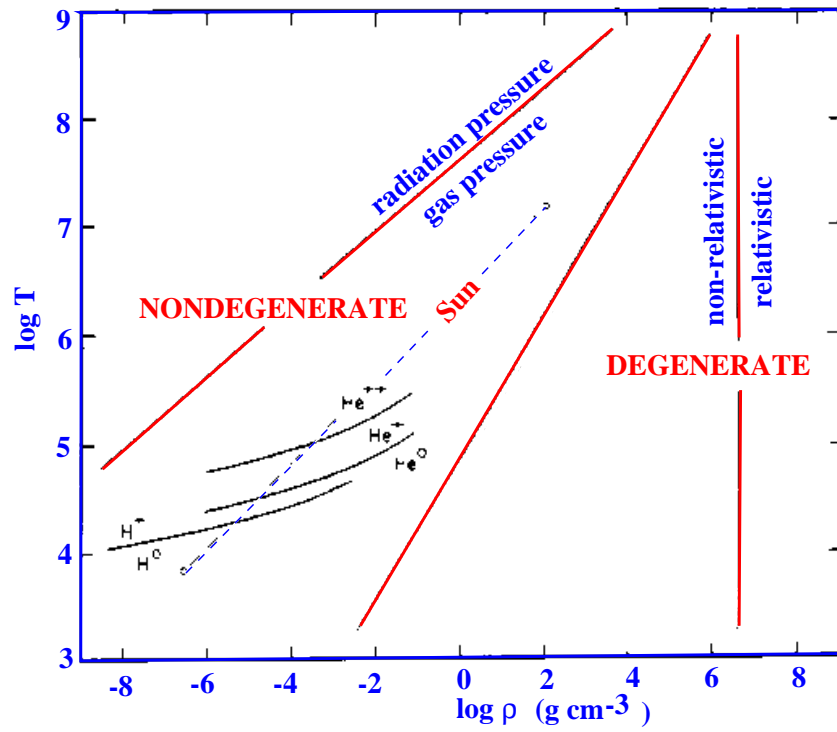
$$= (8\pi c/3h^3) p_0^4/4 = (2\pi c/3h^3) p_0^4$$

$$= (hc/8)(3/\pi)^{1/3} N_e^{4/3}.$$

- Also $N_e = (X + Y/2 + Z/2) \rho/m_H = (1/2)(1 + X) \rho/m_H$.
- For intermediate regions use the full relativistic expression for $v(p)$.
- For ions we may continue to use the non-degenerate equation:
- $P_{\text{ions}} = (1/\mu_{\text{ions}})(\rho kT/m_H)$ where $(1/\mu_{\text{ions}}) = X + Y/4$.

Conditions where degeneracy is important:

- (a) *Non-relativistic* – interiors of *white dwarfs*; *degenerate cores of red giants*.
- (b) *Relativistic* - very high densities only; interiors of *white dwarfs*.



Temperature-density diagram for the equation of state
(Schwarzschild 1958)

THE OPACITY

The rate at which energy flows by radiative transfer is determined by the opacity.

$$dT/dr = -3\kappa L\rho/(16\pi acr^2 T^3) \quad (4)$$

In degenerate stars a similar equation applies with the opacity representing resistance to energy transfer by electron conduction.

Sources of stellar opacity:

1. *bound-bound absorption* (negligible in interiors)

2. *bound-free absorption*

3. *free-free absorption*

4. *scattering by free electrons*

- usually use a mean opacity averaged over frequency, *Rosseland mean opacity* (see textbooks).

Approximate analytical forms for opacity:

High temperature: $\kappa = \kappa_1 = 0.020 \text{ m}^2 \text{ kg}^{-1} (1 + X)$

Intermediate temperature: $\kappa = \kappa_2 \rho T^{-3.5}$ (*Kramer's law*)

Low temperature: $\kappa = \kappa_3 \rho^{1/2} T^4$

- $\kappa_1, \kappa_2, \kappa_3$ are constant for stars of given chemical composition but all depend on composition.

Opacities

