C1: Astrophysics Major Option
Problem Set 5: Advanced Stellar Astrophysics
(Ph. Podsiadlowski, HT06)

1 The Last Stable Circular Orbit [25 points]

In General Relativity, the equation for the radial coordinate r of a test particle orbiting a non-rotating black hole of mass M can be written as

$$\frac{1}{2}\dot{r}^2 + \frac{1}{2}\left(1 - \frac{2GM}{c^2 r}\right)\left(\frac{L^2}{r^2} + c^2\right) = \frac{1}{2}\frac{E^2}{c^2}, \tag{1}$$

where $\dot{r} = \mathrm{d}r/\mathrm{d}t$ and L and E are the angular momentum per unit rest mass and the energy per unit rest mass of the particle, respectively (the particle is assumed to have non-zero rest mass). This equation resembles the energy conservation equation in Newtonian dynamics, $E_\mathrm{N} = 1/2\,\dot{r}^2 + V_\mathrm{eff}(r)$, except for the additional term $-GML^2/c^2\,r^3$ in the effective potential V_eff that becomes dominant at small radii.

a) Treating the problem like a Newtonian one, sketch the effective potential for a particle near a black hole as a function of radius, both for a small and a large value of L. Characterize the possible types of trajectories/orbits in both cases.

b) Show that for each value of L there are two possible circular orbits

$$r_\pm = \frac{L^2 \pm \left[L^4 - 12G^2M^2L^2/c^2\right]^{1/2}}{2GM}, \tag{2}$$

provided that $L^2 > 12G^2M^2/c^2$.

c) Show that the r_+ solution has a minimum value of $r_+^\mathrm{min} = 6GM/c^2$ and argue that this is a stable orbit (i.e. corresponds to a minimum of the effective potential). What does this imply for the r_- solution?

d) Calculate the energy E of a particle at this innermost stable circular orbit and show that it's binding energy per unit rest mass E_B is

$$E_\mathrm{B} = (1 - (8/9)^{1/2})\,c^2 \simeq 0.06\,c^2.$$

e) Discuss briefly what happens as matter orbiting a black hole in an accretion disc approaches the innermost stable orbit. Compare this case to accretion onto a non-magnetic neutron star.

2 Gamma-Ray Bursts [25 points]

A popular model for long-duration gamma-ray bursts (GRBs) is the collapsar model, in which a massive, relatively rapidly rotating helium (or carbon/oxygen) star collapses to form a compact object (e.g. a black hole) surrounded by a disc of matter at nuclear density. Subsequent accretion from the disc causes the formation of a relativistic jet that penetrates the remaining infalling envelope and generates a gamma-ray burst at a large distance away from the star by internal and/or external shocks.

For this problem consider a collapsing helium star of mass $M_{\rm He} = 10\,M_\odot$ and radius $R_{\rm He} = 5 \times 10^8\,$m where the inital mass of the central black hole is $M^0_{\rm BH} = 2\,M_\odot$.

a) Show that the specific angular momentum of the infalling material has to be larger than $2 \times 10^{12}\,\mathrm{m^2\,s^{-1}}$, the specific angular momentum at the last stable orbit for a $2\,M_\odot$ black hole, so that an accretion disc can form. Estimate the characteristic dynamical timescales both for the inner disc and the collapsing helium star. How do these timescales determine the observable characteristics of GRBs?

b) Assume that an amount of relativistic energy $E = 10^{44}\,$J is injected by the central engine of the GRB, driving an expanding relativistic fireball. Estimate the radius at which the fireball becomes optically thin to MeV gamma rays, i.e. the radius at which the optical depth to pair creation $\gamma\,\gamma \rightleftharpoons e^+ + e^-$ becomes less than 1. [You may assume that the cross section for pair creation is given by the Thomson cross section $\sigma_{\rm T} \simeq 6.6 \times 10^{-29}\,\mathrm{m}^2$ and that the typical photon energy is 1 MeV; argue that the optical depth is then given by $\tau_\gamma \sim n_\gamma\,\sigma_{\rm T}\,R$.]

While massive stars are known to be rapidly rotating on the main sequence, they are believed to be spun down efficiently during their evolution by hydrodynamical and magnetohydrodynamical effects and develop cores that are not rotating sufficiently fast to be consistent with the collapsar model. One way of spinning up a helium star is if it is a member of a close binary where it can be spun up by the tidal interaction with a companion star.

c) Consider a $10\,M_\odot$ helium star in a close orbit with a compact star (most likely a neutron star or a black hole) with an orbital period $P_{\rm orb}$. Assume that the spin angular velocity of the helium star is synchronized with the orbital angular velocity of the binary and that the helium star is in solid body rotation. Estimate the maximum orbital period for which the core is sufficiently rapidly rotating that only the innermost $2\,M_\odot$ can collapse directly, while the rest collapses first into a disc [take the typical radius of the inner $2\,M_\odot$ core as $8 \times 10^7\,$m). [Answer: $\sim 5\,$hr]

d) Sketch briefly the evolutionary path that can lead to the formation of such a system.

3 Mass-Transfer Driving Mechanisms [25 points]

Consider a binary consisting of two stars of mass M_1 and M_2 with an orbital separation A and orbital period P.

a) Show that the total angular momentum of the binary can be written as

$$J = \mu\, A^2 \frac{2\pi}{P} = \mu \sqrt{G\,(M_1 + M_2)\,A},$$

where $\mu \equiv M_1\, M_2/(M_1 + M_2)$ is the reduced mass of the system. Show that for conservative mass transfer (where the total mass and the total angular momentum of the system remains constant), the orbital separation is a minimum when $M_1 = M_2$. Sketch the evolution of A as a function of time assuming that $M_1 > M_2$ initially and that mass is transferred from star 1 to star 2. How does this behaviour of A affect the mass-transfer rate, assuming that star 1 *attempts* to expand at a steady rate?

b) Even in the absence of mass transfer, the orbit of a binary will shrink due to the emission of gravitational waves, which causes the loss of orbital angular momentum at a rate

$$\frac{\mathrm{d}J}{\mathrm{d}t} = -\frac{32}{5} \frac{G^{7/2}}{c^5} \frac{\mu^2 M^{5/2}}{A^{7/2}},$$

where $M = M_1 + M_2$ is the total mass of the binary. Show that this implies that the orbital period decreases as

$$\frac{1}{P}\frac{\mathrm{d}P}{\mathrm{d}t} = -\frac{96}{5}\frac{G^3}{c^5}\frac{M^2\mu}{A^4}.$$

By setting the orbital period decay time ($P/\dot{P}$) equal to the age of the Galaxy ($\sim 10^{10}\,\mathrm{yr}$), determine the maximum separation and hence maximum orbital period for which a binary consisting of (i) two low-mass helium white dwarfs with $M_1 = M_2 = 0.3\,M_\odot$, (ii) two massive carbon/oxygen white dwarfs with $M_1 = M_2 = 1\,M_\odot$ and (iii) two neutron stars with $M_1 = M_2 = 1.4\,M_\odot$ are driven into contact by gravitational wave emission within the age of the Galaxy. Discuss the likely/possible fate of the systems in the three cases.

c*) Assume that star 1 loses mass in a stellar wind at a wind mass-loss rate $\dot{M} = 10^{-10}\,M_\odot\,\mathrm{yr}^{-1}$ and that the wind is magnetically coupled to the spin of star 1 up to a radius $10\,R$ away from the star (where R is the radius of the star). Assume further that due to the tidal interaction with the companion star, the spin of star 1 is synchronized with the orbital period (i.e. $P_{\mathrm{spin}} = P_{\mathrm{orb}}$). Estimate the orbital period decay time ($P/\dot{P}$) due to this *magnetic braking* for a system with $M_1 = M_2 = 1\,M_\odot$ and $A = 3R$. [Hint: what is the specific angular momentum lost in the stellar wind?]

4 Neutron-Star Spin-Up by Wind Accretion [25 points]

Consider a massive X-ray binary consisting of a $20\,M_\odot$ star and a $1.4\,M_\odot$ neutron star in a relatively wide orbit where the radius of the massive star is much smaller than its Roche lobe radius. Assume that the massive star loses mass in a steady spherical wind with a wind mass-loss rate $\dot{M}_{\mathrm{wind}} = 10^{-7}\,M_\odot\,\mathrm{yr}^{-1}$ and a wind velocity $v_{\mathrm{wind}} = 10^3\,\mathrm{km\,s^{-1}}$ and that the neutron star has a magnetic field $B = 10^8\,\mathrm{T}$ (assumed to be dipolar) and is spun up (or spun down!) by accretion of some of the wind material.

a) Explain the meaning of Alfvén radius (r_{Alf}) and Bondi-Hoyle radius (r_{BH}). Calculate r_{BH} for the above system and show that the fraction of the wind that is accreted by the neutron star is given by $(r_{\mathrm{BH}}/2A)^2$ where A is the orbital separation of the binary.

The matter that is accreted by Bondi-Hoyle accretion typically has a specific angular momentum that is 1/4 of the specific Keplerian angular momentum at the Bondi-Hoyle radius.

b) Estimate the characteristic size of a disc that would form around the neutron star in the absence of a magnetic field.

c*) Now considering the actual magnetic field of the pulsar, estimate the maximum orbital separation A for which a disc can form around the pulsar (i.e. for which the accreted specific angular momentum is larger than the Keplerian specific angular momentum at the Alfvén radius). Determine the equilibrium spin period of the pulsar for this case.