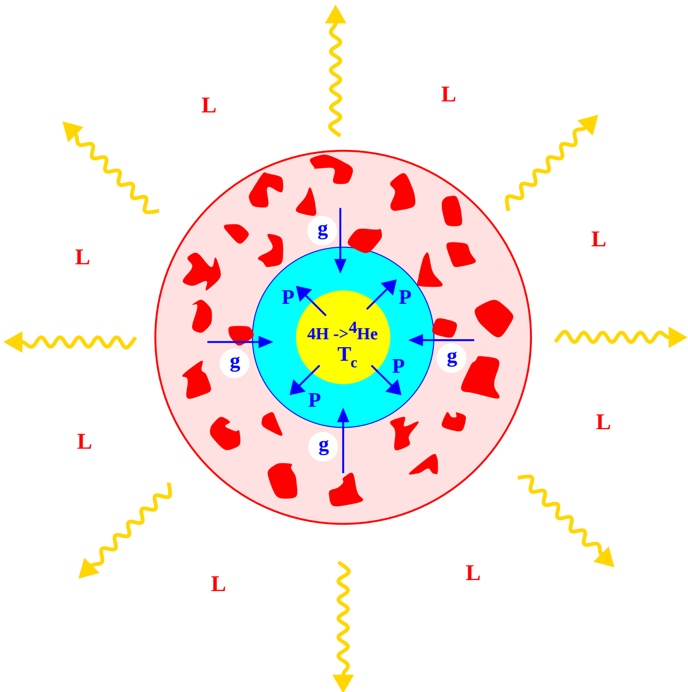
The Structure and Evolution of Stars

Recommended Background Reading

- *Short Option S26 - Stars and Galaxies,* Dr Andrew Bunker
  http://www.physics.ox.ac.uk/users/bunker/s26StarsGalaxies.htm
- *and/or Concepts in Thermal Physics, Chapters 35/36, Stars, White Dwarfs,* Blundell & Blundell



FUNDAMENTAL PRINCIPLES

- Stars are *self-gravitating* bodies in *dynamical equilibrium* $\rightarrow$ *balance of gravity and internal pressure forces* (hydrostatic equilibrium);
- stars lose energy by *radiation from the surface* $\rightarrow$ stars supported by thermal pressure require an *energy source* to avoid collapse, e.g. *nuclear energy, gravitational energy* (energy equation);
- the *temperature structure* is largely determined by the mechanisms by which *energy* is *transported* from the core to the surface, *radiation, convection, conduction* (energy transport equation);
- the *central temperature* is determined by the *characteristic temperature* for the appropriate *nuclear fusion reactions* (e.g. H-burning: 10^7 K; He-burning: 10^8 K);
- normal stars have a *negative 'heat capacity'* (virial theorem): they heat up when their total energy decreases ($\rightarrow$ normal stars contract and heat up when there is no nuclear energy source);
- *nuclear burning is self-regulating* in non-degenerate cores (virial theorem): e.g. a sudden increase in nuclear burning causes expansion and cooling of the core: *negative feedback* $\rightarrow$ *stable nuclear burning;*
- the *global structure* of a star is determined by the *simultaneous satisfaction* of these principles $\rightarrow$ the *local properties* of a star are determined by the *global structure.*
  (Mathematically: it requires the simultaneous solution of a set of coupled, non-linear differential equations with mixed boundary conditions.)

Equation of Stellar Structure

Equation of Hydrostatic Equilibrium:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM_r\rho}{r^2}, \tag{1}$$

Equation of Mass Conservation:

$$\frac{\mathrm{d}M_r}{\mathrm{d}r} = 4\pi r^2\rho, \tag{2}$$

Energy Conservation (Nuclear Plus Gravitational Energy):

$$\frac{\mathrm{d}L_r}{\mathrm{d}r} = 4\pi r^2\rho\left(\varepsilon_r - T\frac{\mathrm{d}S}{\mathrm{d}t}\right), \tag{3}$$

Energy Transport (Radiative Diffusion Equation):

$$L_r = -4\pi r^2\,\frac{4ac}{3\kappa\rho}T^3\,\frac{\mathrm{d}T}{\mathrm{d}r}, \tag{4}$$

Energy Transport by Convection, Convective Stability:

$$\frac{\mathrm{d}T}{\mathrm{d}r} = \frac{\gamma-1}{\gamma}\,\frac{T}{P}\frac{\mathrm{d}P}{\mathrm{d}r}. \tag{5}$$

These equations have to be supplemented by constitutive relations, the equation of state $P(\rho, T)$, the opacity $\kappa(\rho, T)$ and the energy generation rates $\varepsilon_r(\rho, T, X_i)$, and 4 boundary conditions (two at the surface: $L = 4\pi R^2\sigma T_{\mathrm{eff}}^4$ and $P = 2/3g/\kappa$; and two at the centre: $M(r) = 0$ and $L(r) = 0$) to produce a total of 7 equations for 7 unknown variables (P, ρ, T, $M(r)$, $L(r)$, κ, ε_r) as a function of the independent variable r.

In these equations, M_r, is the mass enclosed in a sphere of radius r, L_r the luminosity (energy/time) flowing through a sphere of radius r; P, T, ρ and S, the pressure, temperature, density and entropy of the material, γ the adiabatic exponent.

For material obeying the ideal gas equation $P = (\rho/\mu m_{\mathrm{H}})\,kT$, one can estimate the central temperature, T_{c}, of a self-gravitating object, using eq. 1, as

$$kT_{\mathrm{c}} \simeq \frac{GM\mu m_{\mathrm{H}}}{R}, \tag{6}$$

where M and R are the mass and the radius of the object (for the Sun: $(T_{\mathrm{c}})_\odot \simeq 2\times10^7\,\mathrm{K}$).
($1M_\odot = 1.99\times10^{30}$ kg; $1R_\odot = 6.96\times10^8$ m; $1L_\odot = 3.86\times10^{26}$ W.)

The Virial Theorem

Multiplying the equation of hydrostatic equilibrium

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM_r\rho}{r^2}$$

by $4\pi r^3 \mathrm{d}r$ and integrating the resulting equation from the centre to the surface, one obtains after integrating the l.h.s. by parts

$$4\pi \left[r^3 P\right]_{r=0,P=P_\mathrm{c}}^{r=R,P=P_\mathrm{s}\simeq 0} - 3\int_0^R P\,4\pi r^2\,\mathrm{d}r = -\int_0^R \overbrace{\frac{GM_r}{r}\underbrace{4\pi r^2\rho\,\mathrm{d}r}_{\mathrm{d}M_r}}^{\frac{GM_r\,\mathrm{d}M_r}{r}}.$$

The first term (the boundary term) vanishes both at the center (exactly) and at the surface (approximately), while the last term just gives the total gravitational potential energy, Ω, of the star. Using the relationship between the internal energy (per unit volume) and the pressure for an ideal gas with adiabatic exponent γ,

$$u = \frac{P}{\gamma - 1},$$

the integral on the left becomes

$$\Rightarrow \int_0^R P_r\,4\pi r^2\,\mathrm{d}r = (\gamma - 1)\underbrace{\int_0^R u\,4\pi r^2\,\mathrm{d}r}_{U},$$

were U is the total thermal energy of the star. Combining these equations, one obtains the desired result, *the virial theorem:*

$$\underline{\underline{\Omega + 3(\gamma - 1)\,U = 0}}. \qquad (7)$$

For $\gamma = 5/3$, $\Omega = -2U$, and the total energy, $E = \Omega + U$, can be written as

$$E = -U \sim -\frac{3}{2}N\,\langle kT\rangle$$

(in a qualitative sense). The fact that $E \propto -\langle T\rangle$ implies that, if a star loses energy (e.g. by radiation from the surface), E becomes more negative and the average temperature increases, i.e. *a star losing energy heats up,* which in a thermodynamical sense corresponds to a *negative heat capacity*. This can easily be understood from the fact that, according to the virial theorem, half of the gravitational potential energy released if the star contracts is converted into thermal energy, while the other half is lost from the star. An entirely analogous example is the decay of the orbit of a satellite, where the velocity (and kinetic energy) of the satellite increases as the orbit decays (due to the virial theorem in classical mechanics).

The virial theorem is one of the most important theorems for understanding the structure and evolution of stars.

The Stability of Nuclear Burning

Consider the core of a star burning nuclear fuel at a particular equilibrium rate, ϵ. Assume that this burning rate is perturbed by some amount $\Delta\epsilon$. This increases the energy production in the core and causes an expansion of the core, but that expansion, according to the virial theorem, reduces U and hence T ($\Delta\Omega = -2\Delta U$, i.e. *cools* the core). Since the burning rate is strongly temperature dependent, this will reduce the burning rate, i.e. provide a *strong negative feedback*, making nuclear burning in non-degenerate stars stable.

Comment: This negative feedback does not work if the matter is very degenerate, when the pressure is independent of the temperature. In this case, the ignition of nuclear fuel does not cause an expansion of the core; since the release of nuclear energy then raises the temperature, this will further increase the nuclear burning rate (i.e. now provide a positive feedback); this leads to a *thermonuclear runaway* which may lead to the complete disruption of the star, as in certain supernovae, or at least a hydrodynamical flash, as in the case of the *helium flash*.

The Virial Theorem as a Driving Force of Stellar Evolution

In phases where there is no nuclear burning, the total energy of a star necessarily decreases because energy is radiatated away from the surface. Since $E = 1/2\,\Omega$, Ω must decrease (become more negative), which means that the star contracts (the energy source in this phase is "gravitational energy"). According to the virial theorem, half of the gravitational potential energy released in the contraction is converted into thermal energy, i.e. heats the star. This will continue till the core of the star reaches a temperature that is high enough for particular nuclear reactions to start. The lifecycle of a star is governed by the alternation of such contraction and nuclear burning phases, starting on the pre-main-sequence where a star contracts till its centre reaches the temperature for hydrogen burning ($\sim 10^7$ K). After the core has exhausted its hydrogen, it starts to contract again either until it becomes degenerate or until the central temperature reaches the characteristic temperature for helium burning ($\sim 10^8$ K). Dependent on the mass of the star, these successive contraction and burning phases may continue until the core is completely composed of iron. Since iron has the highest binding energy/baryon, no more nuclear energy can be released and the core has to contract/collapse, producing a *core-collapse supernova.*

In the case where $\gamma \to 4/3$, the virial theorem states that $\Omega = -U \quad \Rightarrow \quad \underline{\underline{E = \Omega + U \approx 0}}$, i.e. the total energy approaches 0, which means that such a star will be strongly affected by small perturbations. Generally, objects become dynamically unstable when $\gamma = 4/3$, which can lead either to the collapse or the complete disruption of the object (depending on the perturbation). In the most massive stars, γ approaches $4/3$, which makes them very susceptible to mass loss.

Energy transport

The size of the *energy flux* is determined by the mechanism that provides the *energy transport: conduction, convection or radiation.* For all these mechanisms the *temperature gradient determines the flux.*

- *Conduction* does not contribute seriously to energy transport through the interior
  - ▷ At high gas density, mean free path for particles << mean free path for photons.
  - ▷ Special case, *degenerate matter* – very effective conduction by electrons.
- The *thermal radiation field* in the interior of a star consists mainly of *X-ray photons in thermal equilibrium with particles.*
- Stellar material is *opaque to X-rays* (bound-free absorption by inner electrons)
- *mean free path for X-rays* in solar interior $\sim 1\,\mathrm{cm}$.
- Photons reach the surface by a *"random walk"* process and as a result of many interactions with matter are degraded from X-ray to optical frequencies.
- After N steps of size l, the distribution has spread to $\simeq \sqrt{N}\,l$. For a photon to "random walk" a distance R_s, requires a *diffusion time* (in steps of size l)

$$t_\mathrm{diff} = N \times \frac{l}{c} \simeq \frac{R_\mathrm{s}^2}{lc} \tag{8}$$

For $l = 1\,\mathrm{cm}$, $R_\mathrm{s} \sim R_\odot \rightarrow t_\mathrm{diff} \sim 5 \times 10^3\,\mathrm{yr}$.

Energy transport by convection:

- *Convection* occurs in liquids and gases when the temperature gradient exceeds some typical value.
- *Criterion for stability against convection (Schwarzschild criterion)*
- *convective stability:* a fluid element is stable against convection if, after a small displacement, material in the fluid element becomes denser than the ambient medium → fluid elements sinks back: $\boxed{\frac{\rho}{\gamma P}\frac{\mathrm{d}P}{\mathrm{d}r} - \frac{\mathrm{d}\rho}{\mathrm{d}r} > 0}$

Equation of State

Ideal Gas:

$$P = \frac{N}{V}kT = \frac{\rho}{\mu m_\mathrm{H}}kT, \tag{9}$$

where μ is the mean particle mass in units of m_H, i.e.

$$\frac{1}{\mu} = 2X + \frac{3}{4}Y + \frac{1}{2}Z,$$

where X is the mass fraction of hydrogen ($X_\odot \simeq 0.7$), Y the mass fraction of helium $Y_\odot \simeq 0.28$) and Z the mass fraction of heavier elements, referred to as metals ($Z_\odot \simeq 0.02$).

Radiation Pressure:

$$P = \frac{1}{3}aT^4. \tag{10}$$

Electron Degeneracy ($T = 0\,K$):

$$P = K_1\left(\frac{\rho}{\mu_e m_\mathrm{H}}\right)^{5/3}, \tag{11}$$

(non-relativistic degeneracy)

$$P = K_2\left(\frac{\rho}{\mu_e m_\mathrm{H}}\right)^{4/3}. \tag{12}$$

(relativistic degeneracy)

Opacity

Thomson (Electron) Scattering:

$$\kappa = 0.020\,\mathrm{m^2\,kg^{-1}}\,(1 + X), \tag{13}$$

Kramer's Opacity:

$$\kappa \propto \rho\,T^{-3.5}, \tag{14}$$

Low-Temperature Opacity:

$$\kappa \propto \rho^{1/2}\,T^4. \tag{15}$$

NUCLEAR REACTIONS

- *Binding energy* of nucleus with Z protons and N neutrons is:

$$Q(Z,N) = \underbrace{[ZM_{\rm p} + NM_{\rm n} - M(Z,N)]}_{\text{mass defect}}\, c^2. \tag{16}$$

- *Energy release:*

$4\,\mathrm{H}\rightarrow{}^4\mathrm{He}$ $\quad 6.3\times10^{14}\,\mathrm{J\,kg^{-1}} = 0.007\,c^2\ (\varepsilon = 0.007)$

$56\,\mathrm{H}\rightarrow{}^{56}\mathrm{Fe}$ $\quad 7.6\times10^{14}\,\mathrm{J\,kg^{-1}} = 0.0084\,c^2\ (\varepsilon = 0.0084)$

- *H burning* already releases most of the available nuclear binding energy.

Nuclear reaction rates:

$$\boxed{\mathbf{1}} + \boxed{\mathbf{2}} \rightarrow \boxed{1{,}2} + \textbf{Energy}$$

Hydrogen Burning ($T \simeq 10^7$ K)
PPI chain:

1. $^1\mathrm{H} + {}^1\mathrm{H} \rightarrow {}^2\mathrm{D} + \mathrm{e}^+ + \nu \qquad +1.44\,\mathrm{MeV}$
2. $^2\mathrm{D} + {}^1\mathrm{H} \rightarrow {}^3\mathrm{He} + \gamma \qquad +5.49\,\mathrm{MeV}$
3. $^3\mathrm{He} + {}^3\mathrm{He} \rightarrow {}^4\mathrm{He} + {}^1\mathrm{H} + {}^1\mathrm{H} \qquad +12.85\,\mathrm{MeV}$

- for each conversion of $^4\mathrm{H} \rightarrow^4$ He, reactions (1) and (2) have to occur twice, reaction (3) once
- the *neutrino* in (1) carries away 0.26 MeV leaving 26.2 MeV to contribute to the luminosity
- reaction (1) is a *weak interaction* $\rightarrow$ *bottleneck* of the reaction chain
- *Typical reaction times* for $T = 2\times10^7$ K are
  (1) 14×10^9 yr
  (2) 6 s
  (3) 10^6 yr
  - ▹ (these depend also on ρ, X_1 and X_2).
  - ▹ *Deuterium* is burned up very rapidly.
- *(very approximate)*

$$\varepsilon_{\rm PP} \propto \rho\, X_{\rm H}^2\, T^4. \tag{17}$$

THE CNO CYCLE $\qquad (T < 10^8\,\mathrm{K})$

- *Carbon, nitrogen and oxygen* serve as *catalysts* for the *conversion of H to He*

$$\begin{aligned}
{}^{12}\mathrm{C} + {}^1\mathrm{H} &\rightarrow {}^{13}\mathrm{N} + \gamma \\
{}^{13}\mathrm{N} &\rightarrow {}^{13}\mathrm{C} + \mathrm{e}^+ + \nu \\
{}^{13}\mathrm{C} + {}^1\mathrm{H} &\rightarrow {}^{14}\mathrm{N} + \gamma \\
{}^{14}\mathrm{N} + {}^1\mathrm{H} &\rightarrow {}^{15}\mathrm{O} + \gamma \\
{}^{15}\mathrm{O} &\rightarrow {}^{15}\mathrm{N} + \mathrm{e}^+ + \nu \\
{}^{15}\mathrm{N} + {}^1\mathrm{H} &\rightarrow {}^{12}\mathrm{C} + {}^4\mathrm{He}
\end{aligned}$$

- The *seed nuclei* are believed to be predominantly ^{12}C and ^{16}O: these are the main *products of He burning*, a later stage of nucleosynthesis.
- *(very approximate)*

$$\varepsilon_{CNO} \propto \rho\, X_{\rm H}\, X_{\rm CNO}\, T^{20}. \tag{18}$$

Helium Burning ($T \simeq 10^8$ K)
Triple α reaction: $^4\mathrm{He} + {}^4\mathrm{He} + {}^4\mathrm{He} \rightarrow {}^{12}\mathrm{C} + \gamma$

- *(very approximate)*

$$\varepsilon_{3\alpha} \propto X_{\rm He}^3\, \rho^2\, T^{30}. \tag{19}$$

White Dwarfs (WDs)

- CO white dwarfs are the remnants of low-/intermediate-mass stars ($M \lesssim 7\,M_\odot$) that lose their envelopes as asymptotic giants (ejecting a planetary nebula in the process)
- they are completely supported by electron-degeneracy pressure (Pauli exclusion principle)

  Mass–Radius Relation for White Dwarfs (non-relativistic):

  $$R \propto \frac{1}{m_e} (\mu_e m_\mathrm{H})^{5/3} M^{-1/3}, \tag{20}$$

  note the inverse mass–radius relationship
- as electrons become relativistic (with increasing mass), WDs can no longer be supported by electron degeneracy $\rightarrow$ maximum mass for WDs

  Chandrasekhar Mass for White Dwarfs:

  $$M_\mathrm{Ch} = 1.457 \left(\frac{2}{\mu_e}\right)^2 M_\odot. \tag{21}$$

Stellar Timescales

Dynamical Timescale:

$$t_\mathrm{dyn} \simeq \frac{1}{\sqrt{4G\rho}}, \tag{22}$$
$$\sim 30\,\mathrm{min}\,\left(\rho/1000\,\mathrm{kg\,m^{-3}}\right)^{-1/2},$$

Thermal (Kelvin-Helmholtz) Timescale:

$$t_\mathrm{KH} \simeq \frac{GM^2}{2RL}, \tag{23}$$

$$\sim 1.5 \times 10^7\,\mathrm{yr}\,(M/M_\odot)^2\,(R/R_\odot)^{-1}\,(L/L_\odot)^{-1},$$

Nuclear Timescale:

$$t_\mathrm{nuc} \simeq M_c/M\;\eta\;(Mc^2)/L, \tag{24}$$
$$\sim 10^{10}\,\mathrm{yr}\,(M/M_\odot)^{-3},$$

(Radiative) Diffusion Timescale:

$$t_\mathrm{diff} = N \times \frac{l}{c} \simeq \frac{R_s^2}{lc}. \tag{25}$$

Derived Relations

Mass–Luminosity Relation (for stars $\sim 1\,M_\odot$):

$$L \simeq L_\odot \left(\frac{M}{M_\odot}\right)^4, \tag{26}$$

Mass–MS Lifetime Relation (for stars $\sim 1\,M_\odot$):

$$T_\mathrm{MS} \simeq 10^{10}\,\mathrm{yr} \left(\frac{M}{M_\odot}\right)^{-3}, \tag{27}$$

Mass–Radius Relation for White Dwarfs (non-relativistic):

$$R \propto \frac{1}{m_e} (\mu_e m_\mathrm{H})^{5/3} M^{-1/3}, \tag{28}$$

Chandrasekhar Mass for White Dwarfs:

$$M_\mathrm{Ch} = 1.457 \left(\frac{2}{\mu_e}\right)^2 M_\odot, \tag{29}$$

Schwarzschild Radius (Event Horizon) for Black Holes:

$$R_\mathrm{S} = \frac{2GM}{c^2} \simeq 3\,\mathrm{km} \left(\frac{M}{M_\odot}\right). \tag{30}$$

Miscellaneous Equations

Distance Modulus:

$$(m - M)_V = 5\,\log\,(D/10\mathrm{pc}), \tag{31}$$

Absolute V Magnitude:

$$M_V = -2.5\,\log\,L/L_\odot + 4.72 + B.C. + A_V, \tag{32}$$

Salpeter Initial Mass Function (IMF):

$$f(M)\,\mathrm{d}M \propto M^{-2.35}\,\mathrm{d}M, \tag{33}$$

Black-Body Relation:

$$L = 4\pi\,R_s^2\,\sigma T_\mathrm{eff}^4, \tag{34}$$

Kepler's Law:

$$a^3 \left(\frac{2\pi}{P}\right)^2 = G(M_1 + M_2). \tag{35}$$

Evolution in the Hertzsprung-Russell (H-R) Diagram

The evolution of low-mass stars can be qualitatively discussed using the H-R diagram of a globular cluster (log luminosity versus log effective temperature; note that the temperature scale is inverted). In this case, one can assume that all stars have approximately the same age. Since the evolutionary timescale scales roughly like M^{-3}, stars of slightly different masses correspond to stars in slightly different evolutionary phases. Hence the H-R diagram, despite the fact that it represents a snapshot at a given time, illustrates all the evolutionary phases a star near the turn-off mass (see below) will pass through.

- *Region A* represents the main sequence in the cluster. All stars in this region are in the core hydrogen-burning phase, the longest evolutionary phase of a star. Stars at lower luminosity along the main sequence have lower masses (where roughly $M \propto L^4$).
- *Turning Point X:* stars near the turning point have (almost) completely consumed the hydrogen in the core and are about to develop first an isothermal, then an electron-degenerate core and are about to leave the main sequence (moving toward lower temperatures, i.e. region B). Note that there are a few stars on the extension of the main sequence beyond the turning point. These are called *blue stragglers* and are slightly more massive stars that have not yet evolved off the main sequence. They most likely represent the products of the interaction with a companion star or are the result of a direct collision and merger with another star in the cluster (a common occurrence in globular clusters).
- *Regions B and C:* Once stars have left the main sequence, they become bigger and more luminous and become subgiants (region B) and ultimately giants (region C). In the giant phase, they evolve at almost constant effective temperature (on so-called Hayashi tracks; there are no hydrostatic equilibrium solutions at lower temperatures). In this phase, stars have a degenerate, very compact core of $\sim 0.01\,R_\odot$ and are surrounded by a convective envelope which fills most of the volume of the star (of $R \sim 20 - 200\,R_\odot$). Hydrogen burning occurs in a shell just outside the core. In region C, the mass of the degenerate hydrogen-exhausted core grows from a mass of around $0.1\,M_\odot$ to a mass of $\sim 0.48\,M_\odot$.
- *Turning Point Y:* When the star reaches the critical mass of about $0.48\,M_\odot$, helium ignites in the centre of the star. Since the core is degenerate, helium ignition is quite explosive leading to a rapid adjustment of the whole structure of the star (so-called *helium flash*). However, the flash is not sufficiently explosive to completely disrupt the star (like in a supernova). Instead the star re-adjusts and quickly settles on the horizontal branch (region D). Therefore, the helium flash marks a temporary peak in the luminosity of the star, defining the *tip of the red-giant branch*.
- *Region D:* After re-establishing hydrostatic and thermal equilibrium, the star spends a significant fraction of its life on the horizontal branch, where it burns helium in the centre (at a temperature $\sim 10^8\,$K), surrounded by a hydrogen-burning shell (the latter is often the dominant nuclear-burning source). After having consumed all the helium in the core, the star again becomes a giant, where the track asymptotically approaches the first giant branch. In this *asymptotic giant phase*, the star has a degenerate carbon/oxygen core surrounded by a helium-rich shell and a hydrogen-rich envelope. The nuclear energy source is hydrogen and helium burning in thin shells surrounding the degenerate core. At some point the star will lose all of its remaining envelope, uncovering the degenerate core which then cools down and becomes a white dwarf (with a size of $\sim 0.01\,R_\odot$).

The energy released in the fusion of 4 protons to one alpha particle is equal to the rest mass energy released in the reaction, i.e. $\Delta E = (4m_{\rm H} - m_{\rm He})\,c^2$. Defining an *efficiency*, η, for this nuclear reaction from

$$\Delta E = \eta \Delta M_{\rm H}\, c^2, \tag{36}$$

where $\Delta M_{\rm H}$ is the hydrogen rest mass consumed, the efficiency can be estimated from

$$\eta = \frac{4m_{\rm H} - m_{\rm He}}{4m_{\rm H}} \simeq 0.007,$$

using $m_{\rm H} = 1.0078\,$a.m.u and $m_{\rm He} = 4.0026\,$a.m.u. Assuming that a star exhausts hydrogen in the innermost $10\,\%$ of its mass on the main sequence and taking an initial hydrogen mass fraction $X = 0.7$, the total nuclear energy produced on the main sequence can be estimated as

$$E_{\rm MS} = \eta \times X \times 0.1 \times M\, c^2$$

On the other hand, the mass–luminosity relation for solar-type stars is approximately given by

$$L = 1\,L_\odot \left(\frac{M}{M_\odot}\right)^4.$$

Dividing $E_{\rm MS}$ by L and eliminating M in favour of L using the L–M relation, then leads to an estimate of the main-sequence lifetime as function of the final luminosity on the main sequence, i.e. the turnoff luminosity, $L_{\rm TO}$,

$$T_{\rm MS} \simeq 10^{10}\,{\rm yr} \left(\frac{L_{\rm TO}}{L_\odot}\right)^{-3/4}. \tag{37}$$

In this particular cluster, the turnoff luminosity is around $1\,L_\odot$, and hence a rough estimate of the age would be $10^{10}\,$yr (i.e. the same as the main-sequence lifetime of a $1\,M_\odot$ star). Note that this estimate ignores metallicity effects (lower-metallicity stars have higher luminosity and shorter main-sequence lifetimes).